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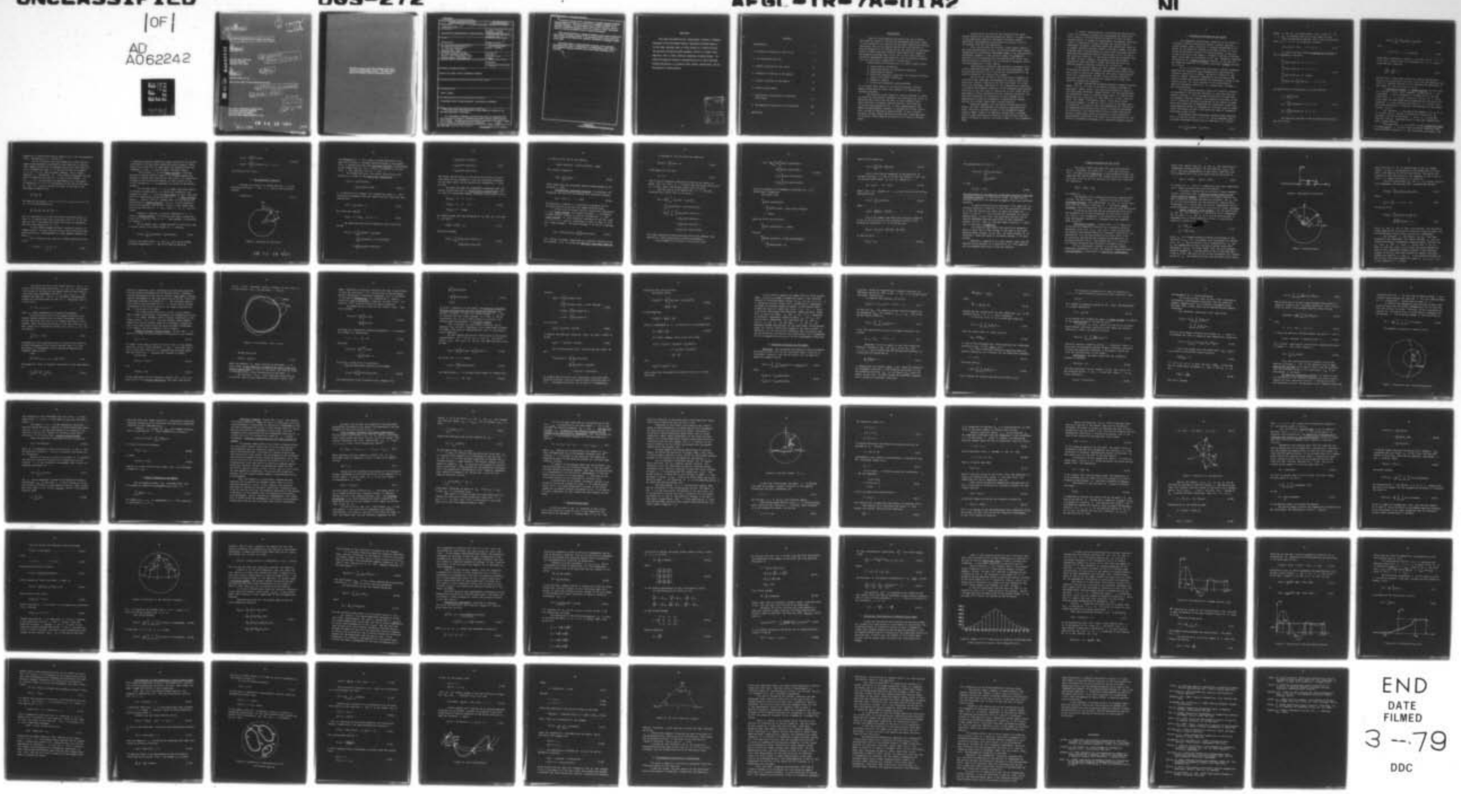

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STATISTICAL FOUNDATIONS OF COLLOCATION

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Helmut Moritz

The Ohio State University
Research Foundation
Columbus, Ohio 43212

14 DGS-272
SCIENTIFIC-16

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Jun 1978

12 81 p.

Scientific Report No. 16

Approved for public release; distribution unlimited

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16 7600 17 03

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFGL-TR-78-0182	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) STATISTICAL FOUNDATIONS OF COLLOCATION		5. TYPE OF REPORT & PERIOD COVERED Scientific. Interim. Scientific Report No. 16
		6. PERFORMING ORG. REPORT NUMBER Report No. 272
7. AUTHOR(s) Helmut Moritz		8. CONTRACT OR GRANT NUMBER(s) F19628-76-C-0010
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Geodetic Science The Ohio State University Columbus, Ohio 43210		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62101F - 760003AG
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01731 Contract Monitor - Bela Szabo/LW		12. REPORT DATE June 1978
		13. NUMBER OF PAGES 75 pages
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

TECH, OTHER

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gravitational Field, Gravity Anomalies, Least-Squares Collocation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper deals with mathematical models suitable as a basis for the statistical treatment of collocation.

As a preparation, stochastic processes on the circle are discussed first; such processes are simple to understand and exhibit already essential features of the problem. Then the paper treats stochastic processes on the sphere, which may be suitable as statistical models for collocation. (con't)

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Lauritzen's theorem on the nonexistence of ergodic Gaussian stochastic process models for collocation is seen to be essentially dependent on the Gaussian character. Two non-Gaussian ergodic models are given, one of a genuinely probabilistic character similar to Lauritzen's model, and another based on a formal probability theory in rotation group space.

This second model gives a statistical foundation of the usual homogeneous and isotropic covariance analysis of the anomalous gravity field; it also provides a basis for the study of the statistical distribution of quantities related to this field.

This model allows a formal statistical treatment of the anomalous gravitational field which is independent of an interpretation of this field as some genuinely physical stochastic process and seems, therefore, to be preferable.

FOREWORD

This report was prepared by Dr. Helmut Moritz, Professor, Technische Hochschule in Graz and Adjunct Professor, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. F19628-76-C-0010, The Ohio State University Research Foundation, Project No. 710334, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, L. G. Hanscom Field, Bedford, Massachusetts, with Mr. Bela Szabo/LW, Project Scientist.

CONTENTS

Introduction

Users of least-squares collocation ask for a theory that gives an answer to practically meaningful questions: What is the accuracy of our results? Can we apply statistical testing techniques? How can we compute statistical distributions of gravity anomalies or of deflections of the vertical? A reasonable answer to these questions requires some statistical theory of the anomalous gravitational field. But is this field really a stochastic phenomenon? Such questions seem to motivate research into the statistical foundations of collocation.

Least-squares collocation has its roots in many fields:

1. Least-squares estimation;
2. Prediction theory of stochastic processes;
3. Approximation theory;
4. Functional analysis, especially the theory of Hilbert spaces with kernel functions;
5. Potential theory;
6. Inverse and improperly posed problems.

All of these "many facets of collocation" present relevant aspects which must be taken into account in a complete and balanced treatment.

The relation to the theory of inverse problems is clear: our data are functionally related to the gravitational field; to determine this field from the data, we must somehow invert those functional relations. Now the gravity field requires infinitely many parameters for its full determination; the number of measurements, however, is essentially finite. Therefore, we have an improperly posed problem. To get a unique solution, we must impose additional conditions, which may have the form of a least-squares principle or of a norm in Hilbert space.

Historically, collocation has developed from least-squares prediction of gravity anomalies, which is an application of the prediction theory of stochastic processes. Hence, statistical considerations have played an essential role in collocation from the very beginning.

Also, the relation to classical least-squares adjustment has soon been noted. In fact, collocation models bear formal resemblance to conventional adjustment models. The characteristic difference, however, is the infinite number of parameters necessary to fully characterize the gravitational field. This fact furnishes an essential link to stochastic processes and to infinite-dimensional Hilbert spaces.

Least-squares estimation and stochastic processes give a very convenient mathematical formalism and terminology. They also provide the basis for a statistical interpretation of the results, essential for feasibility studies.

The practical success of the statistical treatment of collocation has sometimes overshadowed its equally significant analytical aspects, especially the fact that there is a clean analytical structure underlying it. This mathematical structure is based on the harmonic character of the anomalous gravitational field and on the fact that all quantities of this field can be expressed as linear functionals of the anomalous potential. The analytical character of collocation is best brought out by approaching it from the standpoint of approximation theory, working in a Hilbert space with a kernel function.

These two aspects, the statistical and the analytical aspect, are both indispensable and mutually complement each other. This fact, evident already in the fundamental paper (Krarup, 1969), seems to be generally agreed upon, although there is some controversy on details, as may be seen from the papers collected in (Moritz and Sünkel, 1978); cf. also (Dermanis, 1976).

A literal interpretation of the anomalous gravitational field as a stochastic process has encountered two objections. First, there is only one Earth; a probability space of many possible earths is logically unobjectable, but appears unnatural, since all realizations except one (the real Earth) are unobservable. Secondly, Lauritzen (1973) has proved that there is no ergodic Gaussian process, harmonic outside a sphere. This has sometimes been misinterpreted as a proof that no ergodic process modelling the anomalous gravity field exists at all, so that the covariance function, in principle, cannot be estimated from the data. In fact, however, the Gaussian structure enters essentially into Lauritzen's proof, and there do exist non-Gaussian ergodic processes suitable for collocation.

In the present report we shall attempt an elementary discussion of possible stochastic processes on the sphere which are suited as statistical models for the earth's gravitational field. As a preparation, we shall first consider stochastic processes on the circle, which are simpler and already show essential theoretical features.

We shall present two different ergodic stochastic process models. One is, in a way, a non-Gaussian ergodic analogue of Lauritzen's model; there is an underlying probability space of infinitely many different "sample earths". For the second model, the probability space is rotation group space; all realizations differ only by a rotation, so that there is, in fact, only one Earth. This model is extremely simple: it has been called "trivially ergodic" in (Moritz, 1973, p.70). At the same time, it expresses, in a natural way, the homogeneity and isotropy of the anomalous gravitation field, which is usually presupposed in collocation. This second model allows a formal statistical analysis (covariances and distributions) of the terrestrial gravitational field even if we reject the interpretation of this field as a stochastic phenomenon in a genuinely physical sense.

1. Stochastic Processes on the Circle

The anomalous gravitational potential of the earth is a harmonic function outside the earth's surface. Outside a certain sphere, such a function is uniquely determined by its values on the spherical surface: from these values it is obtained by solving an exterior Dirichlet problem. To any continuous function on the sphere, a harmonic function in outside space can be made to correspond in this way. Instead of studying the behavior of a spatial harmonic function, we may thus investigate the behavior of a (rather arbitrary) surface function on a sphere.

It is in this sense that the earth's external anomalous potential has frequently been mathematically described by a stochastic process on a sphere. The earth's surface is very nearly a sphere. Therefore, homogeneity and isotropy on the earth's surface may approximately be formulated in terms of the rotation group. This also accounts for the usefulness of spherical harmonics, which are, in a natural way, related to the rotation group.

For the present purpose, almost all essential features are preserved if we consider functions in the plane that are harmonic outside a circle instead of functions in space that are harmonic outside a sphere, and stochastic processes on the circle instead of stochastic processes on the sphere. Furthermore, this reduction of dimensionality essentially simplifies the problem and makes its mathematical structure easy to understand. Therefore, we shall start with the study of stochastic processes on the circle.

A continuous and continuously differentiable function $f(t)$ on the unit circle $0 \leq t < 2\pi$ can be expanded into a uniformly convergent Fourier series (Smirnow, v.II, p.417):

$$f(t) = \sum_{k=0}^{\infty} (a_k \cos kt + b_k \sin kt) , \qquad (1\text{-}1)$$

where a_k and b_k are coefficients; since $\sin kt = 0$ for $k = 0$, b_o is arbitrary and will be put equal to zero.

This representation defines $f(t)$ also for arbitrary real t $(-\infty < t < \infty)$ as a periodic function:

$$f(t \pm 2k\pi) = f(t), \quad k = 1, 2, 3, \ldots \tag{1-2}$$

; In view of the well-known <u>orthogonality relations</u> of the trigonometric functions:

$$\begin{aligned}
&\int_0^{2\pi} \cos kt \cos \ell t \, dt = 0 \quad \text{if} \quad k \neq \ell, \\
&\int_0^{2\pi} \sin kt \sin \ell t \, dt = 0 \quad \text{if} \quad k \neq \ell, \\
&\int_0^{2\pi} \cos kt \sin \ell t \, dt = 0 \quad \text{always}, \\
&\int_0^{2\pi} \cos^2 kt \, dt = \int_0^{2\pi} \sin^2 kt \, dt = \pi \quad \text{if} \quad k > 0,
\end{aligned} \tag{1-3}$$

the coefficients of the series (1-1) are given by

$$\begin{aligned}
a_o &= \frac{1}{2\pi} \int_0^{2\pi} f(t) dt, \\
a_k &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt \, dt \quad \text{if} \quad k > 0, \\
b_k &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt \, dt \quad \text{if} \quad k > 0.
\end{aligned} \tag{1-4}$$

The function, defined in the xy-plane outside and on the unit circle,

$$f(x,y) = \sum_{k=0}^{\infty} r^{-k}(a_k \cos kt + b_k \sin kt) \tag{1-5}$$

with

$$r = \sqrt{x^2 + y^2} \;, \quad t = \arctan \frac{y}{x} \tag{1-6}$$

being polar coordinates, reduces on the unit circle $r = 1$ to (1-1) and is readily seen to be harmonic for $r > 1$, satisfying Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \;. \tag{1-7}$$

We thus have a very simple one-to-one relation between the function (1-1) defined on the unit circle and the harmonic function (1-5) defined outside; it will therefore be sufficient in the sequel to limit our study to (1-1).

A stochastic process, or random function, on the circle is a function $f(t,\omega)$ which depends, in addition to t, on a parameter ω which represents a "random choice". For any fixed value $\omega = \omega_1$ we get a function $f(t,\omega_1)$ of t only, which under the above-mentioned assumptions has the form (1-1); different ω_1 give different functions of t of form (1-1), which are considered as different "realizations" of the random process $f(t,\omega)$.

For instance, ω may denote the numbers 1, 2, 3, 4, 5, 6, so that $f(t,\omega)$ denotes 6 functions of form (1-1). By throwing a die we can determine ω (e.g., $\omega_1 = 5$) and the function $f(t,\omega_1)$ associated with it; this will explain the term, random function.

More generally, ω is a point in some probability space, or sample space, Ω. In this space we define a measure, such that measurable subsets of Ω are associated with events, the

measure of a subset denoting the probability of the corresponding event. The measure of Ω itself is 1.

Let us illustrate this well-known fact, which can be found in any textbook on probability (my favorite is (Feller, 1957, 1966)) by means of the example just given, the throw of a die. Probability space Ω is the set of the six integers $\{1, 2, 3, 4, 5, 6\}$. Any of these integers, say 4, forms a subset of Ω, denoted by $\{4\}$. This subset corresponds to the event of throwing the face "4". To each of the subsets $\{1\}$, $\{2\}, \ldots, \{6\}$ we associate the same measure 1/6. The event of throwing a "2" or a "4" corresponds to the sum of the sets $\{2\}$ and $\{4\}$ and has probability equal to the sum of the individual probabilities:

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3} .$$

The event of throwing a "1" or a "2" or a "3" or a "4" or a "5" or a "6" has the probability

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1 ,$$

that is, certainly, as it must be from an intuitive point of view: it is certain that one of the faces from "1" to "6" will show up. This illustrates the intuitive reason for demanding that the total measure of Ω is 1.

In this simple example we have 6 possible choices, or "sample points". In more relevant case we need infinitely many possible choices, corresponding to a more general probability space Ω.

Let us return to our case of a random function on the circle

$$f = f(t,\omega) , \quad 0 \leq t < 2\pi , \quad \omega \in \Omega , \tag{1-8}$$

Ω denoting a general probability space, which will be specialized later on. To get these simple but basic concepts firmly fixed in our mind, let us state again the meaning of the two arguments t and ω, using slightly different terms.

The variable t is the "space variable", defining position in actual physical space. This becomes immediately evident on taking into account that the circle is a simplified analogue to the terrestrial sphere, so that a point on the circle, defined by t, corresponds to a point on the earth's surface.

On the other hand, ω, so to speak, describes chance: it defines a random choice. In statistical mechanics, the probability space Ω is called phase space; we shall sometimes find this terminology convenient and call ω a phase variable. Anyway, ω serves a kind of "random label" to distinguish one realization (or sample function) $f(t,\omega_1)$ of our stochastic process from another realization $f(t,\omega_2)$, both sample functions being functions of t only, since ω_1 or ω_2 are constants.

Generally speaking, a quantity depending on ω is called a random variable. This explains the name, random function, for a function $f(t,\omega)$ of t that depends, in addition, on "chance" ω.

Let us expand such a random function on the circle into a Fourier series (1-1) with respect to t. We have

$$f(t,\omega) = \sum_{k=0}^{\infty}\left[a_k(\omega)\cos kt + b_k(\omega)\sin kt\right] ; \qquad (1\text{-}9)$$

clearly, the coefficients a_k and b_k will now be random. variables depending on ω. By (1-4) they are given by

$$a_0(\omega) = \frac{1}{2\pi}\int_0^{2\pi} f(t,\omega)dt \;,$$

$$a_k(\omega) = \frac{1}{\pi}\int_0^{2\pi} f(t,\omega)\cos kt \; dt \;, \quad k > 0 \;, \tag{1-10}$$

and similarly for $b_k(\omega)$.

2. The Covariance Function

Consider the values of a random function f at two different positions, t and $t+s$ (Fig.1) and form their product:

$$f(t)f(t+s) \;; \tag{2-1}$$

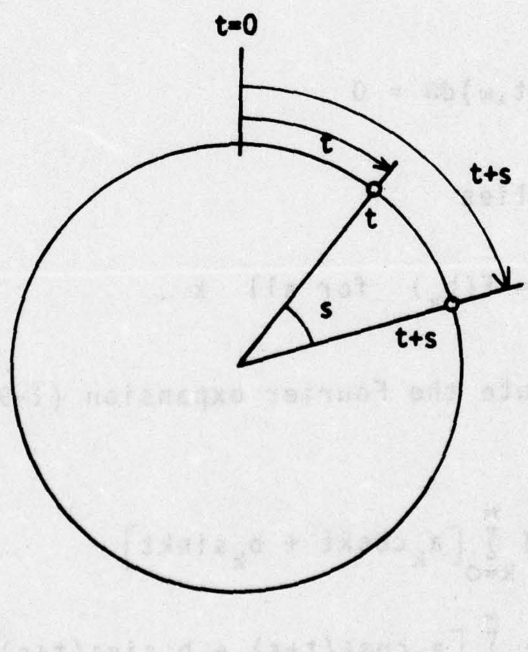

Figure 1. Positions on the circle

the dependence on ω will always be understood even if not explicitly written. A suitably defined average of the product (2-1) is nothing else than the <u>covariance function</u> corresponding to the random function $f(t) = f(t,\omega)$; it depends on the distance s and, possibly, also on t and ω .

For random functions, the natural definition of the average is in terms of the <u>statistical expectation</u> E :

$$\begin{aligned} C(s,t) &= E\{f(t)f(t+s)\} \\ &= \int_\Omega f(t,\omega)f(t+s,\omega)d\Omega \ ; \end{aligned} \tag{2-2}$$

E is defined as an integral over probability space Ω . This definition presupposes that the random function itself has zero expectation:

$$E\{f\} = \int_\Omega f(t,\omega)d\Omega = 0 \ . \tag{2-3}$$

By (1-10) this implies

$$E\{a_k\} = 0 = E\{b_k\} \quad \text{for all} \quad k \ . \tag{2-4}$$

We substitute the Fourier expansion (1-9) into (2-2) and get

$$\begin{aligned} C(s,t) &= E\{ \sum_{k=0}^{\infty} \left[a_k \cos kt + b_k \sin kt\right] \cdot \\ &\quad \cdot \sum_{\ell=0}^{\infty} \left[a_\ell \cos\ell(t+s) + b_\ell \sin\ell(t+s)\right] \} \\ &= E\{\sum_k\sum_\ell [a_k a_\ell \cos kt \cos\ell(t+s) + \end{aligned}$$

$$+ b_k b_\ell \sin kt \sin \ell(t+s) +$$

$$+ a_k b_\ell \cos kt \sin \ell(t+s) + \tag{2-5}$$

$$+ b_k a_\ell \sin kt \cos \ell(t+s)]\} .$$

The formal multiplication of the two Fourier series is justified since, by our assumption, they are uniformly convergent. For the same reason, we can perform the integration E term by term.

We shall now make the fundamental assumption that the Fourier coefficients are all statistically uncorrelated, that is, that all covariances between different coefficients vanish:

$$E\{a_k a_\ell\} = 0 \quad \text{if} \quad k \neq \ell ,$$

$$E\{b_k b_\ell\} = 0 \quad \text{if} \quad k \neq \ell , \tag{2-6}$$

$$E\{a_k b_\ell\} = 0 \quad \text{always} .$$

We further assume that the variances of a_k and b_k, for each k, are equal:

$$E\{a_k^2\} = E\{b_k^2\} = c_k . \tag{2-7}$$

Then (2-5) becomes

$$C(s,t) = \sum_{k=0}^{\infty} \left[E\{a_k^2\} \cos kt \cos k(t+s) + E\{b_k^2\} \sin kt \sin k(t+s)\right] . \tag{2-8}$$

In view of (2-7) and of the identity

$$\cos kt \cos k(t+s) + \sin kt \sin k(t+s) = \cos ks$$

this finally reduces to

$$C(s) = \sum_{k=0}^{\infty} c_k \cos ks \ , \qquad (2\text{-}9)$$

which shows that the covariance function then depends on the distance s only.

The Empirical Covariance Function.- In practice, one frequently has only one realization of a stochastic process

$$f(t) = f(t,\omega) \ , \quad \omega = \text{const.} \qquad (2\text{-}10)$$

The question is whether it is possible to estimate the covariance function using this one sample function only.

In this case we cannot form the statistical expectation E, the "phase average" (if probability space Ω is denoted as phase space); instead, we form an average over t, the "space average" M (for stochastic processes on the real line, $-\infty < t < \infty$, t may be interpreted as time, so that M will be a "time average"). The space average M of (2-1) is defined as

$$\Gamma(s) = M\{f(t)f(t+s)\} = \frac{1}{2\pi}\int_0^{2\pi} f(t)f(t+s)dt \ , \qquad (2\text{-}11)$$

$f(t)$ being, as always, understood as a periodic function (1-2). The function $\Gamma(t)$ is called the empirical covariance function.

In analogy to (2-3) we have the condition

$$M\{f(t)\} = \int_0^{2\pi} f(t)dt = 0 \ , \tag{2-12}$$

which means by (1-4) that

$$a_o = 0 \ . \tag{2-13}$$

This is not an essential restriction since we can always replace $f(t)$ by $f(t) - a_o$, for which the zero-order coefficient is, in fact, zero. Thus we may assume (2-13) to hold.

Then the sum in (1-1) begins with $k = 1$, and substituting this series into (2-11) we get

$$\begin{aligned}
\Gamma(s) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{\infty} \left[a_k \cos kt + b_k \sin kt\right] \cdot \\
&\quad \cdot \sum_{\ell=1}^{\infty} \left[a_\ell \cos \ell(t+s) + b_\ell \sin \ell(t+s)\right] dt \\
&= \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \int_0^{2\pi} \Big[a_k a_\ell \cos kt \ \cos \ell(t+s) + \\
&\qquad + b_k b_\ell \sin kt \ \sin \ell(t+s) + \\
&\qquad + a_k b_\ell \cos kt \ \sin \ell(t+s) + \\
&\qquad + b_k a_\ell \sin kt \ \cos \ell(t+s)\Big] dt \ ,
\end{aligned}$$

the formal operations (series multiplication and termwise integration) are again justified by uniform convergence.

The orthogonality relations (1-3) give at once:

$$\Gamma(s) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\Big[a_k^2\int_0^{2\pi}\cos kt\,\cos k(t+s)dt +$$

$$+ b_k^2\int_0^{2\pi}\sin kt\,\sin k(t+s)dt +$$

$$+ a_k b_k\int_0^{2\pi}\cos kt\,\sin k(t+s)dt + \tag{2-14}$$

$$+ a_k b_k\int_0^{2\pi}\sin kt\,\cos k(t+s)dt\Big] ,$$

since all products of trigonometric functions for $k \neq \ell$ vanish after integration.

We further have

$$\int_0^{2\pi}\cos kt\,\cos k(t+s)dt =$$

$$= \int_0^{2\pi}(\cos^2 kt\,\cos ks - \cos kt\,\sin kt\,\sin ks)dt$$

$$= \pi\cos ks ,$$

again by (1-3), and similarly

$$\int_0^{2\pi}\sin kt\,\sin k(t+s)dt = \pi\cos ks .$$

Finally,

$$\int_0^{2\pi}\big[\cos kt\,\sin k(t+s) + \sin kt\,\cos k(t+s)\big]\,dt =$$

$$= \int_0^{2\pi}\sin k(2t+s)dt = 0 .$$

Hence (2-14) reduces to

$$\Gamma(s) = \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)\cos ks \ . \tag{2-15}$$

This is the Fourier expansion of the empirical covariance function. In view of (2-10), the function $f(t)$ and its Fourier coefficients a_k and b_k depend on ω :

$$a_k = a_k(\omega) \ , \quad b_k = b_k(\omega) \ . \tag{2-16}$$

Hence, also $\Gamma(s)$ depends on ω , so that (2-15) can be written more explicitly:

$$\Gamma(s,\omega) = \sum_{k=1}^{\infty}\gamma_k(\omega)\cos ks \ , \tag{2-17}$$

where

$$\gamma_k(\omega) = \frac{1}{2}\left[a_k^2(\omega) + b_k^2(\omega)\right] \ . \tag{2-18}$$

Let us now compare the empirical covariance function (2-15) or (2-17) with the true covariance function (2-9). Forming the expectation E of (2-18) we have

$$E\{\gamma_k\} = E\{\gamma_k(\omega)\} = \frac{1}{2}E\{a_k^2\} + \frac{1}{2}E\{b_k^2\} \ ,$$

so that by (2-7)

$$E\{\gamma_k\} = c_k \ . \tag{2-19}$$

The expectation of (2-17) is

$$E\{\Gamma(s,\omega)\} = \sum_{k=1}^{\infty} E\{\gamma_k(\omega)\}\cos ks$$

$$= \sum_{k=1}^{\infty} c_k \cos ks \ ,$$

so that

$$E\{\Gamma(s)\} = C(s) \ , \qquad (2\text{-}20)$$

<u>the expectation of the empirical covariance function is the true covariance function</u>. In statistical terms, the empirical covariance function is an unbiased estimate of the true covariance function.

It would be particularly desirable if the empirical covariance function is identical to the true covariance function, or if the two function are equal at least for almost all ω (that is, for all ω with the possible exception of a set of measure zero). In this case, the covariance function can be exactly estimated from one realization of the stochastic process $f(t,\omega)$, that is, from one sample function ω = const. This is the case of <u>ergodicity</u>.

This name has been taken from statistical mechanics, where it means that a time average is the same as the corresponding phase average. In our case, the space average M of $f(t)f(t+s)$ should be equal to the phase average E of this product.

Obviously, ergodicity is a very special case, and the question arises whether it is possible at all. This question will be answered positively in the next section.

3. Ergodic Processes on the Circle

The case in which the empirical covariance function $\overline{C}(s,\omega)$ coincides, for almost all ω, with the true one, $C(s)$, has been called ergodicity in the preceding section. By comparing the coefficients of the respective Fourier expansions (2-9) and (2-15) we get the necessary and sufficient condition for ergodicity:

$$a_k^2(\omega) + b_k^2(\omega) = 2c_k \tag{3-1}$$

for almost all ω.

The meaning of this condition should be carefully kept in mind. The coefficients c_k, defined by (2-6), are given nonrandom constants. The coefficients a_k and b_k on the left-hand side are, however, functions of ω and hence random variables. Thus the condition (3-1) is certainly very restrictive.

It should be recalled that we have derived (3-1) under the assumption of uniform convergence of the Fourier series for $f(t,\omega)$. This assumption is not essential; for a proof under more general conditions (integrability) see (Zygmund, 1968, pp.36-37).

Lauritzen's Theorem.- In particular, it is impossible to satisfy the ergodicity condition by a stochastic process defined by (1-9) with $a_k(\omega)$ and $b_k(\omega)$ being uncorrelated and normally distributed (Gaussian) stochastic variables of zero expectation. This has been proved by Lauritzen (1973, p.65) by explicitly calculating the variance of the empirical covariance function $\overline{C}(t,\omega)$ and showing that it is non-zero. (For ergodic processes this variance is evidently zero.)

For us, Lauritzen's theorem is an obvious, almost elementary consequence of (3-1). For Gaussian random variables, uncorrelatedness is equivalent to statistical independence.

Hence (2-6) implies that all a_k and b_k are statistically independent random variables. If the functions $a_k(\omega)$ and $b_k(\omega)$ can vary independently of each other, then (3-1) can be violated at will. Eq. (3-1) would only be satisfied if

$$a_k(\omega) = \text{const.}, \quad b_k(\omega) = \text{const.} \tag{3-2}$$

for almost all ω, which is incompatible with zero expectation (2-4). These contradictions prove the theorem.

Lauritzen's theorem may be concisely, though somewhat loosely, formulated thus: <u>a Gaussian random process on the circle cannot be ergodic</u>. Looking for an ergodic process, we must, therefore, consider non-Gaussian processes. The a_k and b_k will be uncorrelated, but not necessarily statistically independent. It is known that statistical independence implies uncorrelatedness; the converse is true only for normal processes.

<u>Ergodic Processes: First Example</u>.- Let the coefficients a_k and b_k, for different k, be statistically independent; for the same k, a_k and b_k will only be uncorrelated, in agreement with the third equation of (2-6). To satisfy the ergodicity condition (3-1) we take

$$\begin{aligned} a_k &= \sqrt{2c_k}\,\cos\omega_k\,, \\ b_k &= \sqrt{2c_k}\,\sin\omega_k\,, \end{aligned} \tag{3-3}$$

where ω_k is a random variable uniformly distributed in the interval $0 \leq \omega_k < 2\pi$. This means that the probability density $\phi(\omega_k)$ of ω_k has the form of Fig.2. Geometrically, a_k and b_k are represented in Fig.3. We have a random vector with components (a_k, b_k) of fixed length $\sqrt{2c_k}$ but with randomly variable azimuth ω_k. The end point of this vector thus describes a circle. Any point of the circle corresponds to a random choice of ω_k. The probability that the end point of the vector

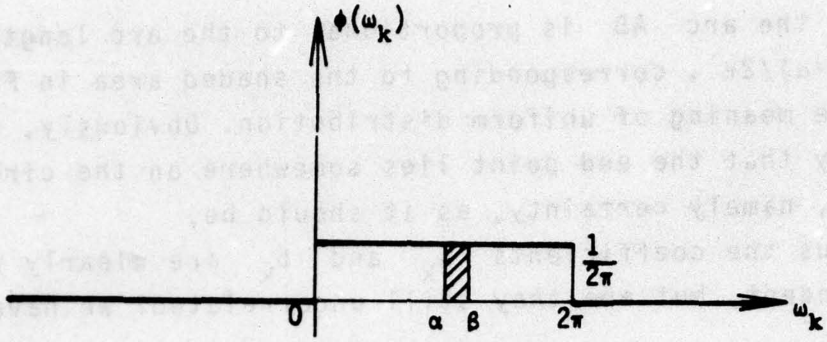

Figure 2. Rectangular distribution

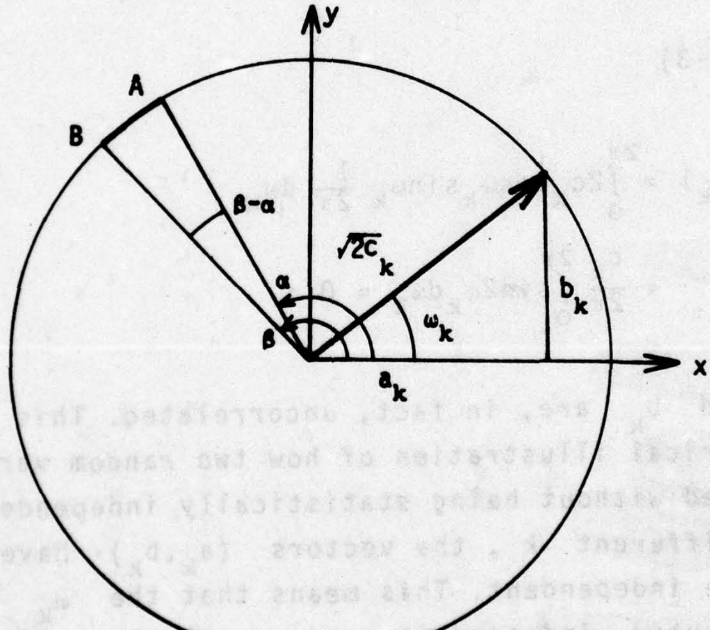

Figure 3. Random azimuth

falls onto the arc AB is proportional to the arc length, namely $(\beta-\alpha)/2\pi$, corresponding to the shaded area in Fig.2; this is the meaning of uniform distribution. Obviously, the probability that the end point lies somewhere on the circle is $2\pi/2\pi = 1$, namely certainty, as it should be.

Thus the coefficients a_k and b_k are clearly statistically dependent, but are they still uncorrelated? We have

$$E\{a_k b_k\} = \int_0^{2\pi} a_k(\omega_k) b_k(\omega_k) \phi_k(\omega_k) d\omega_k \tag{3-4}$$

where

$$\phi_k(\omega) = \frac{1}{2\pi} \quad , \quad 0 \leqq \omega_k < 2\pi \ , \tag{3-5}$$

so that by (3-3)

$$\begin{aligned} E\{a_k b_k\} &= \int_0^{2\pi} 2c_k \cos\omega_k \sin\omega_k \frac{1}{2\pi} d\omega_k \\ &= \frac{c_k}{2\pi} \int_0^{2\pi} \sin 2\omega_k d\omega_k = 0 \ ; \end{aligned} \tag{3-6}$$

hence a_k and b_k are, in fact, uncorrelated. This provides a simple geometrical illustration of how two random variables can be uncorrelated without being statistically independent.

For different k, the vectors (a_k, b_k) have been supposed to be independent. This means that the ω_k are uniformly distributed, independent random variables. Consider two ω_k, say, ω_2 and ω_3. Each ω_k varies from 0 to 2π, that is, over the (unit) circle. Since the joint probability space of two independent random variables is the Cartesian product of the two individual probability spaces, the joint probability space of ω_2 and ω_3 is the Cartesian product of two circles. The joint probability space of $(\omega_1, \omega_2, \dots, \omega_n)$ is the product of n circles.

The Fourier series of the random function $f(t,\omega)$ involves infinitely many coefficients a_k and b_k. The probabilistic event of sorting out one sample function thus requires infinitely many independent choices of $\omega_1, \omega_2, \omega_3, \ldots$ The probability space for $f(t,\omega)$ is, therefore, the Cartesian product of infinitely many circles, or ω represents the infinite vector

$$\omega = (\omega_1, \omega_2, \omega_3, \ldots) , \tag{3-7}$$

each ω_k being independently uniformly distributed.

Finally we prove that if one sample function of our present ergodic process has a uniformly convergent Fourier series, then this will hold for all sample functions of this process. Since the absolute values of sine and cosine are, at most, equal to 1, the Fourier series (1-1), with $a_o = 0$, has the majorant

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|) . \tag{3-8}$$

Convergence of this majorant series is clearly sufficient for uniform convergence of our Fourier series; that it is also necessary is a consequence of the Theorem of Denjoy-Lusin (Zygmund, 1968, p.232).

Since

$$\sqrt{a^2 + b^2} \leqq |a| + |b| \leqq 2\sqrt{a^2 + b^2} , \tag{3-9}$$

convergence of (3-8) is logically equivalent to the convergence of

$$\sum_{k=1}^{\infty} \sqrt{a_k^2 + b_k^2} = \sum_{k=1}^{\infty} \sqrt{2c_k} , \tag{3-10}$$

which is, therefore, also a necessary and sufficient condition for the uniform convergence of our Fourier series. Therefore, uniform convergence of the Fourier series of <u>one</u> sample function implies convergence of the right-hand side of (3-10). Since this right-hand side does not depend on ω, the left-hand side of this equation must converge for all ω, which implies uniform convergence of the Fourier series of the sample functions for any ω, which was to be shown.

The uniform distribution on a circle is even simpler than a normal distribution. Furthermore, the "probability circle" $0 \leqq \omega < 2\pi$ seems, somehow, to be a natural counterpart of the "space circle" $0 \leqq t < 2\pi$. Thus the present simple example seems to be a quite natural model for a stochastic process on the circle, more natural than any Gaussian model; furthermore it is ergodic. The next example is still simpler.

<u>Ergodic Processes: Second Example</u>.- We now take ω itself as a random variable uniformly distributed in the interval

$$0 \leqq \omega < 2\pi \,, \tag{3-11}$$

or, what is the same, on the unit circle. Thus, in the random function $f(t,\omega)$, both variables t and ω now range over a unit circle, the circle for t representing "ordinary space" and the circle for ω representing "probability space".

We now take

$$f(t,\omega) = f(t+\omega) \,. \tag{3-12}$$

Let

$$f(t,0) = f(t) \tag{3-13}$$

be one realization of the stochastic process, for $\omega = 0$; we shall call it the <u>initial realization</u>. Any other realization

$f(t,\omega) = f(t+\omega)$ represents simply a rotation of the circle, or of the function $f(t)$, by the angle ω (Fig.4).

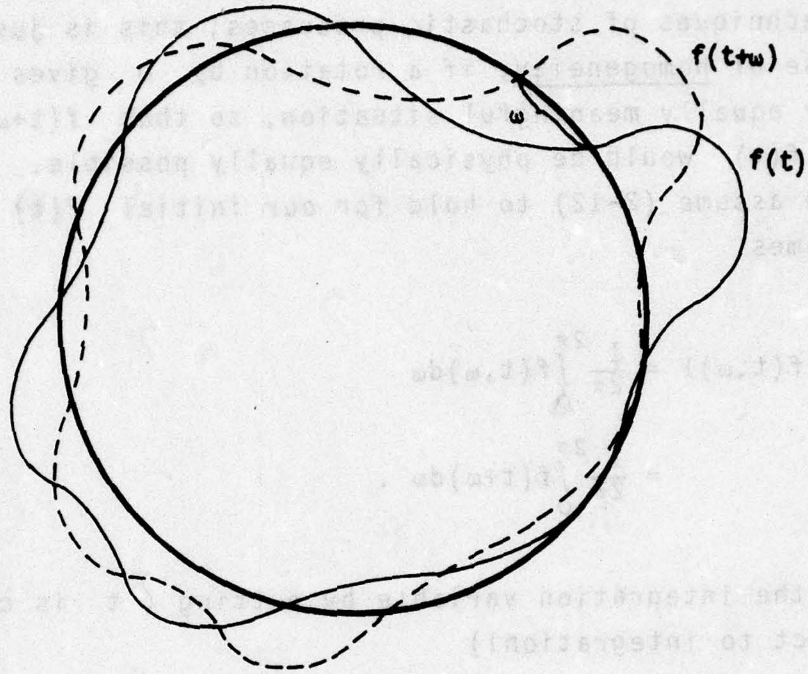

Figure 4. The rotation $f(t) \Rightarrow f(t+\omega)$

We may also write

$$f(t+\omega) = R_\omega f(t) , \qquad (3\text{-}14)$$

where the operator R_ω means rotation by the angle ω. In other terms, <u>we may identify our probability space (3-12) with rotation group space</u>. In fact, in the plane, the rotation group is one-dimensional, being characterized by one angle ω.

The functions $f(t,\omega)$ differ from each other only by a rotation; they are not essentially different (Fig.4). This

model, therefore, is suited to represent the case in which there is only one realization $f(t)$ and we wish to use the mathematical techniques of stochastic processes; this is justified in the case of homogeneity, if a rotation by ω gives a physically equally meaningful situation, so that $f(t+\omega)$ instead of $f(t)$ would be physically equally possible.

We assume (2-12) to hold for our initial $f(t)$. Then (2-3) becomes

$$\begin{aligned} E\{f(t,\omega)\} &= \frac{1}{2\pi}\int_0^{2\pi} f(t,\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t+\omega)\,d\omega \; . \end{aligned} \tag{3-15}$$

We change the integration variable by putting (t is constant with respect to integration!)

$$t + \omega = u \; , \quad d\omega = du \; , \tag{3-16}$$

obtaining

$$\begin{aligned} E\{f(t,\omega)\} &= \frac{1}{2\pi}\int_t^{t+2\pi} f(u)\,du \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(u)\,du = 0 \end{aligned} \tag{3-17}$$

by (2-12), so that (2-3) is satisfied.

Now the covariance function (2-2) becomes

$$C(s,t) = \frac{1}{2\pi}\int_0^{2\pi} f(t+\omega)f(t+s+\omega)\,d\omega \; . \tag{3-18}$$

The substitution (3-16) transforms this integral into

$$\frac{1}{2\pi}\int_0^{2\pi} f(u)f(u+s)du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)f(t+s)dt \tag{3-19}$$

$$= \mathcal{C}(s)$$

by (2-11). Therefore, in this model, the <u>empirical covariance function coincides with the true covariance function</u>; the process is ergodic. In fact, the "phase average" E is seen to coincide with the "space average" M . Since E can be transformed into M by a simple change of variables, the process under consideration is <u>trivially ergodic</u>.

This is obviously a very simple situation, but an important one. We shall, therefore, try to understand it better by studying the spectral representation, that is, the Fourier series expansion.

We shall denote the Fourier coefficients of the initial representation $f(t) = f(t,0)$ by a_k and b_k . The coefficients $a_k(\omega)$ and $b_k(\omega)$ of $f(t,\omega)$ are then given by (1-10). We have

$$a_o(\omega) = \frac{1}{2\pi}\int_0^{2\pi} f(t+\omega)dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = 0 \tag{3-20}$$

by (2-12). For $k > 0$ we get

$$a_k(\omega) = \frac{1}{\pi}\int_0^{2\pi} f(t+\omega)\cos kt\, dt \tag{3-21}$$

and substituting (ω is constant with respect to integration!)

$$t + \omega = v \;, \quad dt = dv \;, \tag{3-22}$$

we have

$$a_k(\omega) = \frac{1}{\pi}\int_0^{2\pi} f(v)\cos k(v-\omega)dv$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(v)(\cos kv \cos k\omega + \sin kv \sin k\omega)dv$$

$$= \cos k\omega \cdot \frac{1}{\pi}\int_0^{2\pi} f(v)\cos kv\, dv + \qquad (3\text{-}23)$$

$$+ \sin k\omega \cdot \frac{1}{\pi}\int_0^{2\pi} f(v)\sin kv\, dv$$

or, by (1-4),

$$a_k(\omega) = a_k \cos k\omega + b_k \sin k\omega \; . \qquad (3\text{-}24)$$

In exactly the same way, replacing $\cos kt$ by $\sin kt = \sin k(v-\omega)$ we get

$$b_k(\omega) = - a_k \sin k\omega + b_k \cos k\omega \; . \qquad (3\text{-}25)$$

Let us now evaluate (2-6), using (3-24) and (3-25). We get

$$E\{a_k(\omega)a_\ell(\omega)\} = \frac{1}{2\pi}\int_0^{2\pi} a_k(\omega)a_\ell(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(a_k \cos k\omega + b_k \sin k\omega) \cdot$$

$$\cdot (a_\ell \cos \ell\omega + b_\ell \sin \ell\omega)d\omega \; .$$

On termwise multiplication and integration, using the orthogonality relations (1-3), we readily obtain the value zero if $k \neq \ell$. Proceeding similarly, we see that all orthogonality

relations (2-6) are satisfied.

We further obtain

$$E\{a_k(\omega)^2\} = \frac{1}{2\pi}\int_0^{2\pi}(a_k\cos k\omega + b_k\sin k\omega)^2 d\omega \tag{3-26}$$
$$= \frac{1}{2}(a_k^2 + b_k^2) .$$

In the same way,

$$E\{b_k^2(\omega)\} = \frac{1}{2}(a_k^2 + b_k^2) , \tag{3-27}$$

which is independent of ω , so that (2-7) is satisfied with

$$c_k = \frac{1}{2}(a_k^2 + b_k^2) . \tag{3-28}$$

We finally compute, using (3-24) and (3-25),

$$a_k(\omega)^2 + b_k(\omega)^2 = (a_k\cos k\omega + b_k\sin k\omega)^2 +$$
$$+ (- a_k\sin k\omega + b_k\cos k\omega)^2$$
$$= a_k^2 + b_k^2 .$$

Thus

$$a_k(\omega)^2 + b_k(\omega)^2 = 2c_k ,$$

which shows that the ergodicity condition (3-1) is, in fact, satisfied.

Let us finally compare this model with our first ergodic model. In the first model, probability space is the Cartesian product of infinitely many circles $0 \leqq \omega_k < 2\pi$ $(k = 1, 2, 3,...)$, ω being the infinite vector (3-7), consisting of independent, uniformly distributed random variables. In the present model, probability space is simply one circle $0 \leqq \omega < 2\pi$, ω being a uniformly distributed one-dimensional random variable. Therefore, in the first model, a_k and a_ℓ for $k \neq \ell$, depending on different independent random variables ω_k and ω_ℓ, are statistically independent. The same holds for a_k and b_ℓ, and for b_k and b_ℓ. For the same k, a_k and b_k are dependent though uncorrelated. On the other hand, in the present model, all Fourier coefficients depend on the same variable ω; therefore, all are statistical dependent, but, as we have seen, any two different coefficients are uncorrelated, as a consequence of the orthogonality relations (1-3) for trigonometric functions.

4. Stochastic Processes on the Sphere

Notations.- Our preceding considerations about stochastic processes on the circle can be translated almost literally to the sphere. Instead of the Fourier series (1-1) we have the spherical-harmonic series

$$f(\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[a_{nm} R_{nm}(\theta,\lambda) + b_{nm} S_{nm}(\theta,\lambda)\right] \tag{4-1}$$

where

$$\begin{aligned} R_{nm}(\theta,\lambda) &= P_{nm}(\cos\theta)\cos m\lambda \;, \\ S_{nm}(\theta,\lambda) &= P_{nm}(\cos\theta)\sin m\lambda \;, \end{aligned} \tag{4-2}$$

$P_{nm}(\cos\theta)$ being the (conventional) Legendre functions (cf. Heiskanen and Moritz, 1967, p.29); n and m are called degree and order, respectively.

To simplify the notation, let us put

$$S_{nm}(\theta,\lambda) = R_{n,-m}(\theta,\lambda) \ , \quad m=1,2,\ldots,n, \tag{4-3}$$

so that any R_{nm} with negative second subscript denotes the corresponding S_{nm}, for instance, $R_{5,-3} = S_{53}$. Then (4-1) may be simply written as

$$f(\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} R_{nm}(\theta,\lambda) \ , \tag{4-4}$$

if for the coefficients we use an analogous notational convention:

$$a_{n,-m} = b_{nm} \ , \quad m=1,2,\ldots,n. \tag{4-5}$$

Sometimes it will be useful to use fully normalized harmonics, denoted by $\bar{R}_{nm}$ and $\bar{S}_{nm}$, or by $\bar{R}_{nm}$ with $-n \leqq m \leqq n$, which differ from the conventional harmonics by a factor and are normalized by

$$\frac{1}{4\pi} \iint_{\sigma} \bar{R}^2_{nm} d\sigma = 1 \ , \tag{4-6}$$

σ denoting the unit sphere (ibid., p.31). Spherical harmonics are orthogonal functions: if we integrate the product of any two different functions R_{nm} (or, of course, $\bar{R}_{nm}$) over the sphere, we get zero. The fully normalized spherical harmonics form a system of orthonormal functions:

$$\overline{M}\{R_{nm}R_{qp}\} = \delta_{nq}\delta_{mp} , \tag{4-7}$$

where

$$\overline{M}\{\cdot\} = \frac{1}{4\pi}\iint_{\sigma}(\cdot)d\sigma \tag{4-8}$$

denotes now the average over the unit sphere and $\delta_{k\ell}$ is the Kronecker delta, 1 if $k = \ell$ and 0 otherwise.

If we write (4-4) in fully normalized harmonics,

$$f(\theta,\lambda) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \overline{a}_{nm}\overline{R}_{nm}(\theta,\lambda) , \tag{4-9}$$

then the coefficients are simply given by

$$\overline{a}_{nm} = \overline{M}\{f\overline{R}_{nm}\} , \tag{4-10}$$

in view of the orthonormality; these equations are a shorthand notation of eqs. (1-76), ibid., p.31.

As a final notational convention regarding spherical harmonic expansions, we introduce the two-dimensional parameter

$$t = (\theta,\lambda) \tag{4-11}$$

and write (4-9) as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \overline{a}_{nm}\overline{R}_{nm}(t) ; \tag{4-12}$$

this stresses the analogy with the case of the circle.

The stochastic parameter will again be denoted by $\omega \in \Omega$, Ω being probability space with total measure 1. Then

$$f(t,\omega)$$

will denote a stochastic process on the sphere. The expectation E is again defined by

$$E\{\cdot\} = \int_{\Omega} (\cdot) d\Omega \tag{4-13}$$

as an average over probability space, or <u>phase average</u>, as opposed to the <u>space average</u> $\overline{M}$ defined by (4-8).

In analogy to (1-5), there is a one-to-one correspondence between continuous functions on the sphere and harmonic functions in space: the spatial function

$$f(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{a_{nm}}{r^{n+1}} R_{nm}(\theta,\lambda) \tag{4-14}$$

satisfies Laplace's equation outside σ . Therefore, there is a one-to-one correspondence between harmonic stochastic processes in space and stochastic processes on the sphere, and we can limit our considerations to the latter.

<u>Covariances</u>.- We again assume that our stochastic process is centered:

$$E\{f(t,\omega)\} = 0 . \tag{4-15}$$

Then the covariance $C(t,u)$ between $f(t,\omega)$ and $f(u,\omega)$ at two different points t and u on the unit sphere σ is, as usual, defined by

$$C(t,u) = E\{f(t)f(u)\} , \tag{4-16}$$

the dependence on ω being understood.

As in the circular case, we shall limit ourselves to continuously differentiable functions. Then the spherical-harmonic expansion will be a uniformly convergent series (Kellogg, 1929, p.259), which can be multiplied and termwise integrated.

We, therefore, substitute (4-12) into (4-16):

$$C(t,u) = E\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{a}_{nm} \bar{R}_{nm}(t) \cdot$$

$$\cdot \sum_{q=0}^{\infty} \sum_{p=-q}^{q} \bar{a}_{qp} \bar{R}_{qp}(u) \} ,$$

multiply and integrate termwise with respect to ω (that is, interchange the order of summation and integration), obtaining

$$C(t,u) = \sum_n \sum_m \sum_q \sum_p E\{\bar{a}_{nm} \bar{a}_{qp}\} \bar{R}_{nm}(t) \bar{R}_{qp}(u) \ . \qquad (4\text{-}17)$$

Let us now assume that the coefficients $\bar{a}_{nm} = \bar{a}_{nm}(\omega)$ are mutually uncorrelated random variables:

$$E\{\bar{a}_{nm} \bar{a}_{qp}\} = 0 \qquad (4\text{-}18)$$

if $q \neq n$ or $p \neq m$ or both, and that $E\{\bar{a}^2_{nm}\}$ is the same for all coefficients of degree n, that is, for all m; we put

$$E\{\bar{a}^2_{nm}\} = \frac{c_n}{2n+1} \ . \qquad (4\text{-}19)$$

Then (4-17) becomes

$$C(t,u) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{c_n}{2n+1} \bar{R}_{nm}(t)\bar{R}_{nm}(u) \ . \tag{4-20}$$

Now we make use of the well-known decomposition formula of spherical harmonics (cf. Heiskanen and Moritz, 1967, p.33,eq. (1-82')), which in our present notation takes the form

$$P_n(\cos\psi) = \frac{1}{2n+1} \sum_{m=-n}^{n} \bar{R}_{nm}(t)\bar{R}_{nm}(u) \ , \tag{4-21}$$

with

$$t = (\theta,\lambda) \quad \text{and} \quad u = (\theta',\lambda') \ , \tag{4-22}$$

ψ being the spherical distance between the points t and u :

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda'-\lambda) \ , \tag{4-23}$$

and $P_n(\cos\psi)$ denoting the (conventional) Legendre polynomial of degree n . Thus (4-20) reduces to

$$C(\psi) = \sum_{n=0}^{\infty} c_n P_n(\cos\psi) \ . \tag{4-24}$$

Thus, the covariance function depends only on the spherical distance ψ . This is the important case of <u>homogeneity and isotropy</u>; it is seen to result from the postulate that the variances (4-19) of all coefficients $\bar{a}_{nm}$ of the same degree n are equal.

<u>The Empirical Covariance Function</u>.- If there is only one realization of the stochastic process, we cannot directly compute the true covariance function C defined by (4-16) and expressed by (4-24). We may again try to compute an empirical

covariance function Γ by replacing the phase average E by a suitable space average and hope that Γ will be a good estimate of C ; if possible, Γ should even be equal to C .

In view of the homogeneity and isotropy, we must integrate not only over the sphere (homogeneity), but in addition over the azimuth (isotropy). Therefore, we must supplement the average $\overline{M}$, defined by (4-8), by additionally averaging over the azimuth α . The resulting average M may be defined by

$$M\{\cdot\} = \frac{1}{8\pi^2} \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} (\cdot)\sin\theta d\theta d\lambda d\alpha \ . \tag{4-25}$$

The geometric situation is shown by Fig.5.

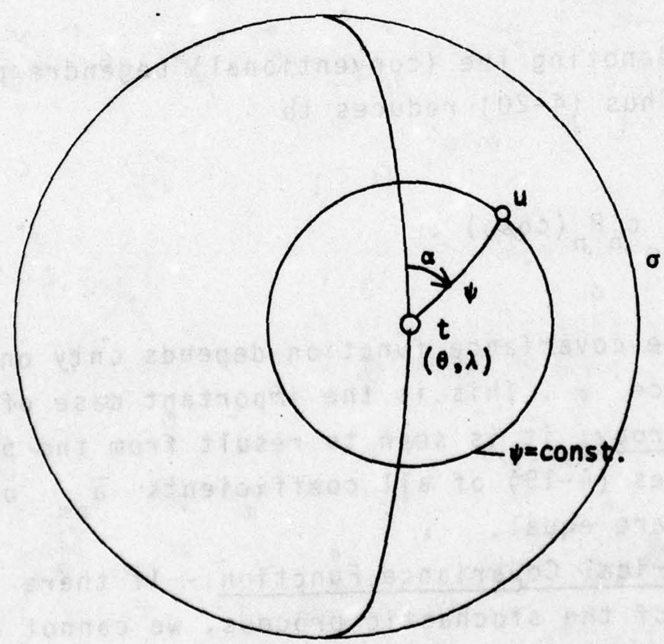

Figure 5. Integration over rotation group space

The averaging is first performed over the circle ψ = const. , whose center $t = (\theta,\lambda)$ is then made to vary over the whole sphere σ .

The angles λ, θ, α can be regarded as the three Eulerian angles defining a rotation in three-dimensional space, that is, λ, θ, α are the coordinates of an element of the rotation group or, of a "point" in "rotation group space". Therefore, M will be called a rotation group average.

Hence the empirical covariance function is given by

$$\Gamma(\psi) = M\{f(t)f(u)\} , \tag{4-26}$$

where M is defined by (4-25) and the points t and u have the spherical distance ψ , which is constant with respect to the integration.

Because of the way in which the average M is computed, the empirical covariance function will depend only on the distance ψ . It can, therefore, be expanded into a series of Legendre polynomials of ψ :

$$\Gamma(\psi) = \sum_{n=0}^{\infty} \gamma_n P_n(\cos\psi) . \tag{4-27}$$

The γ_n can be expressed in terms of the spherical-harmonic coefficients a_{nm} of the same n , in full analogy to (2-18). The derivation is given in (Heiskanen and Moritz, 1967, pp.257-259); the result is eq. (7-28), loc.cit., which in the present notation reads

$$\gamma_n = \sum_{m=-n}^{n} \bar{a}_{nm}^2 . \tag{4-28}$$

Note that this very simple expression is obtained by using conventional harmonics on the left-hand side and fully normalized harmonics on the right-hand side.

Clearly, γ_n , as well as $\bar{a}_{nm}$, are random variables, that is, functions of ω . Their expectation is given by

$$E\{\gamma_n\} = E\{\gamma_n(\omega)\} = \sum_{m=-n}^{n} E\{\bar{a}^2_{nm}(\omega)\} .$$

In view of (4-19) this becomes

$$E\{\gamma_n\} = c_n , \tag{4-29}$$

so that

$$E\{\Gamma(\psi)\} = C(\psi) , \tag{4-30}$$

exactly as in the circular case (2-20): $\Gamma(\psi)$ is an unbiased estimate of $C(\psi)$.

5. Ergodic Processes on the Sphere

For an ergodic process, $\Gamma(\psi)$ coincides with $C(\psi)$. The ergodicity condition, corresponding to (3-1), is

$$\sum_{m=-n}^{n} \bar{a}^2_{nm}(\omega) = c_n , \tag{5-1}$$

for almost all ω ; c_n is independent of ω . This condition is equivalent to $\gamma_n = c_n$.

Lauritzen's Theorem.- Assume that $\bar{a}_{nm}(\omega)$ are normally distributed (Gaussian) random variables. For Gaussian variables, uncorrelatedness is equivalent to statistical independence. From our basic presupposition (4-18) it thus follows that the $\bar{a}_{nm}(\omega)$ must be statistically independent of each other. Then the summands on the right-hand side are independent functions of ω, so that (5-1) will be violated for almost all ω. Loosely formulated: a Gaussian random process on the sphere cannot be ergodic.

The present simple proof of Lauritzen's theorem suffers from the slight logical defect that (5-1) has been derived on the assumption that our stochastic process is sufficiently smooth (differentiable). Since Gaussian random variables may take arbitrarily large values, the convergence of the corresponding spherical-harmonic series cannot be guaranteed; still less are we sure that the corresponding realizations will all be differentiable (this may even be a practical argument against admitting a Gaussian process as mathematical model for the terrestrial gravity field). In fact, (5-1), just as (3-1), holds for more generally assumptions, but we have not proved this because, for the present ergodic models, differentiability can be presupposed.

Thus, our deduction of Lauritzen's theorem has the character of a plausibility argument rather than of a fully rigorous mathematical proof. It has, however, the decisive advantage of showing the essential statistical situation underlying it, and the fact that the Gaussian character of the process is essential to the theorem: only for Gaussian distributions, uncorrelatedness implies statistical independence. This simple fact is hidden below the mathematical intricacies of Lauritzen's (1973,p.65) proof and has not always been clearly understood.

We shall now consider two examples of (non-Gaussian) stochastical processes on the sphere, corresponding to the two examples for the circle given in sec. 3.

First Example: Uniformly Distributed Coefficients.- The two Fourier coefficients a_k and b_k define a two-dimensional vector whose end point lies on a circle of radius $\sqrt{2c_k}$ (Fig.3). Similarly, the 2n+1 coefficients a_{nm} (n fixed, $-n \leqq m \leqq n$) form a (2n+1)-dimensional vector

$$\underline{a} = \left[a_{n,-n}\, ,\, a_{n,-n+1}\, ,\, \ldots\, ,\, a_{n,n-1}\, ,\, a_{n,n}\right] \qquad (5\text{-}2)$$

whose end point lies on a sphere of radius $\sqrt{c_n}$ in R_{2n+1} (Euclidean space of dimension 2n+1); in fact, (5-1) may be written

$$|\underline{a}|^2 = c_n \,. \qquad (5\text{-}3)$$

Assume now that different realizations of the stochastic process correspond to different positions of the endpoint of $\underline{a}$ on this sphere. In other terms, if $\underline{e}$ is the unit vector corresponding to $\underline{a}$, then

$$\underline{a}(\omega) = \sqrt{c_n}\, \underline{e}(\omega)\ , \qquad (5\text{-}4)$$

the unit vector being a function of ω : the random vector $\underline{a}$ has a random direction but a constant length, in complete analogy to (3-3). The random directions $\underline{e}(\omega)$ are uniformly distributed in our (2n+1)-dimensional space: probability is given by an area on the unit sphere in this space; cf.(Feller, 1967, p. 68) for R_3 .

In view of (5-3), the 2n+1 coefficients a_{nm} of the same degree n are statistically dependent, but they are uncorrelated: it is easy to see that (4-18) holds for them. In fact, this means that, for two different components of the

vector $\underline{a}$ or of the vector $\underline{e}$, say e_i and e_j, the integral, over the unit sphere σ_{2n} in R_{2n+1}, of its product $e_i e_j$ is zero:

$$\int_{\sigma_{2n}} e_i e_j d\sigma_{2n} = 0 . \tag{5-5}$$

Denote the left-hand side of this equation by Q_{ij}:

$$Q_{ij} = \int_{\sigma_{2n}} e_i e_j d\sigma_{2n} ; \tag{5-6}$$

we must prove that Q_{ij} is zero.

In fact, if follows from the definition (5-6) that Q_{ij} is invariant with respect to an interchange of the two axes x_i and x_j; it is thus the same regardless of whether the coordinate system is right-handed or left-handed. Because of the spherical symmetry, Q_{ij} is invariant with respect to rotation and to reflection; it only depends on the geometrical configuration. This geometrical configuration--2n+1 mutually orthogonal axes--remains unchanged if we replace the x_j-axis by its opposite direction, giving

$$\int_{\sigma_{2n}} e_i(-e_j) d\sigma_{2n} = -Q_{ij} ,$$

which must, therefore, be equal to Q_{ij}. From $Q_{ij} = -Q_{ij}$ we get $Q_{ij} = 0$ and hence (5-5).

The reader is invited to make this reasoning clear to himself for the case of three-dimensional space with $i = 1$ and $j = 2$. (In this case, (5-5) is equivalent to the orthogonality of the first-degree harmonics. Why?)

So far, we have restricted our considerations to the $2n+1$ a_{nm} corresponding to the same degree n. Let us now consider two different degrees, say n and n'. Any two coefficients a_{nm} belonging to two different degrees will be assumed to be stochastically independent. Thus the probability space Ω is the Cartesian product of infinitely many unit spheres:

$$\Omega = \sigma_2 \times \sigma_4 \times \sigma_6 \times \sigma_8 \times \ldots \times \sigma_{2n} \times \sigma_{2n+2} \times \ldots \quad (5\text{-}7)$$

where σ_{2n} denotes the 2n-dimensional unit sphere in R_{2n+1}. Thus the dimensionality of the spheres increases with increasing n, in contrast to the spherical case where the probability space is the Cartesian product of infinitely many identical circles.

To repeat: in the present model any two different a_{nm} are uncorrelated, but for different reasons: if the two coefficients belong to different degrees, then they are uncorrelated as a consequence of the statistical independence; if the two coefficients belong to the same degree n, then they are uncorrelated because of the orthogonality relation (5-5).

A second model of an ergodic stochastic process is obtained by taking the probability space Ω as rotation group space; this is the three-dimensional analogue of the second example of an ergodic process considered in sec. 3. In view of its basic importance we shall devote the next section to this model.

6. Rotation Group Space

As we have seen in sec. 3, rotations of the circle, which constitute the rotation group in two dimensions, are described by one parameter ω ranging from 0 to 2π : the

group of rotations of the plane forms a one-dimensional space, which may be identified with the unit circle.

Rotations of the sphere, which make up the rotation group in three dimensions, are described by three parameters, for which we may take three Eulerian angles: the group of rotations of three-dimensional space forms itself a three-dimensional space, whose coordinates are the three Eulerian angles. This three-dimensional rotation group space cannot be identified with the unit sphere. This is in contrast to the case of rotations of the circle and accounts for the greater complexity of the present case.

Various authors use various definitions of Eulerian angles. We follow the definition of Synge (1960, p.18), which is fairly widely used and is best suited for the present purpose because of its relation to the spherical coordinates θ, λ .

Let a rectangular coordinate system XYZ be rotated into a position xyz by a general spatial rotation. This rotation is split up into three successive rotations around coordinate axes. The first rotation is about the Z-axis through an angle Λ ; it transforms the XYZ-system into X_1Y_1Z . The second rotation is about the Y_1-axis through an angle Θ ; thus we obtain a system $X_2Y_1Z_2$. Finally we rotate about the Z_2-axis about an angle Ψ in a positive, or $-\Psi$ in the negative, sense, to to obtain the desired system xyz .

The three angles Λ, Θ, Ψ are the Euler angles. They may be illustrated in the following way (Fig.6). The angles Φ and Λ are the usual polar coordinates of the new z-axis. Let Q be the point in which the z-axis intersects the unit sphere, and denote by $\bar{x}$ and $\bar{y}$ the parallels through Q to the x- and y-axis, respectively. Then Ψ is the angle which $\bar{x}$ forms with the meridian, positive when counted counterclockwise (the figure shows a negative Ψ).

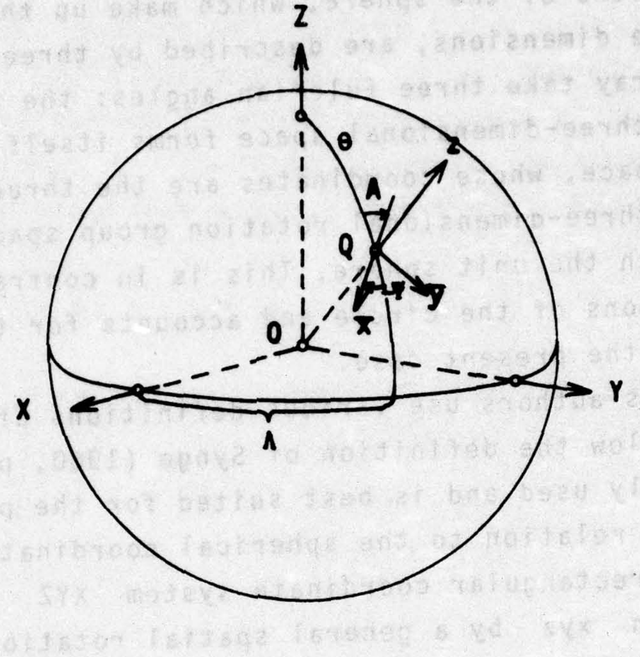

Figure 6. The Euler angles Λ, Θ, Ψ

In the usual terminology, the angle $-\Psi$ is nothing else than the azimuth, counted clockwise, of the negative $\overline{x}$-direction. We shall, therefore, put

$$A = -\Psi \tag{6-1}$$

and consider Θ, Λ, A as our final Eulerian angles.

These three angles define a point ω in rotation group space, which we shall denote by Ω (we shall later interpret it as our probability space); we thus put

$$\omega = (\Theta, \Lambda, A) \; . \tag{6-2}$$

The respective ranges are

$$0 \leqq \Theta \leqq \pi ,$$

$$0 \leqq \Lambda < 2\pi , \qquad (6-3)$$

$$0 \leqq A < 2\pi .$$

A rotation defined by the three Euler angles (6-2) will be denoted by R_ω . The value

$$\omega_o = (0, 0, 0) \qquad (6-4)$$

corresponds to the identity transformation I, leaving the axes XYZ unchanged; symbolically,

$$R_o = I . \qquad (6-5)$$

A unit vector t defined by spherical coordinates θ , λ has the components

$$t = \begin{bmatrix} \sin\theta \cos\lambda \\ \sin\theta \sin\lambda \\ \cos\theta \end{bmatrix} . \qquad (6-6)$$

It will be symbolically abbreviated as

$$t = (\theta,\lambda) ; \qquad (6-7)$$

this notation has already been used before; cf.equation (4-11).

The rotation (6-2) transforms the vector t into another unit vector, which we shall denote by

$$R_\omega t ; \qquad (6-8)$$

it is convenient to consider R_ω as a rotation matrix, so that (6-8) is the usual product of a matrix and a vector.

Clearly, the Euler angles Θ, Λ of the rotation R_ω are completely different from and independent of the coordinates θ, λ of the vector t. There is, however, an interesting relation between these two sets of quantities. Form the triple

$$\tau = (\theta, \lambda, \alpha) , \tag{6-9}$$

with an arbitrary value α between 0 and 2π, and

$$- \tau = (- \theta, - \lambda, - \alpha) . \tag{6-10}$$

Then it is easily seen that

$$R_{-\tau} t = e_Z , \tag{6-11}$$

which is the unit vector of the Z-axis. Thus, the operation $R_{-\tau}$ rotates an arbitrary unit vector t into the Z-axis. This simple fact will be of importance later on.

After these introductory geometrical considerations we are in a position to construct our stochastic process. We take a basic function

$$f(t) = f(\theta, \lambda) \tag{6-12}$$

on the unit sphere and define our stochastic process by

$$f(t,\omega) = f(R_\omega t) . \tag{6-13}$$

This is in analogy to the two-dimensional case, equations (3-12) and (3-14); we shall also follow the respective developments in sec. 3 as closely as possible.

Again, the functions $f(t,\omega)$ differ from each other only by a rotation of the sphere; they are not essentially different. Our model is suited to represent the case in which we have only one realization $f(t)$ but wish to formally use the mathematical techniques of stochastic processes. This is the case of the terrestrial gravitational field, where

$$f(t) = T(\theta,\lambda) \tag{6-14}$$

is the anomalous potential at sea level. The choice (6-13) is intimately connected with homogeneity and isotropy, i.e., with invariance of essential features with respect to rotations R_ω. More about this will be said in the next section.

According to (6-13), probability space Ω is rotation group space, a point $\omega \in \Omega$ being defined by the three Eulerian angles (6-2). The expectation

$$E\{\cdot\} = \iiint\limits_{\Omega}(\cdot)\,d\Omega \tag{6-15}$$

is an integral over rotation group space. The integration is to be extended over the range (6-3); the problem is to find a suitable volume element $d\Omega$, defining a probability measure.

The product of two rotations R is again a rotation. The vector

$$R_1 R_2 t \tag{6-16}$$

is obtained by rotating the vector first by the matrix R_2 and then by the matrix R_1. Assume that a spherical triangle $P_1P_2P_3$ is brought by a rotation R_1 into the position $P_1'P_2'P_3'$; the configuration (angles and sides) of both triangles is obviously identical. Let t_1, t_2, t_3 be the position vectors of P_1, P_2, P_3; all are, of course, unit vectors. Similarly t_1', t_2', t_3' are defined (Fig.7). Then

$$t_1' = R_1 t_1 \;, \quad t_2' = R_1 t_2 \;, \quad t_3' = R_1 t_3 \;. \tag{6-17}$$

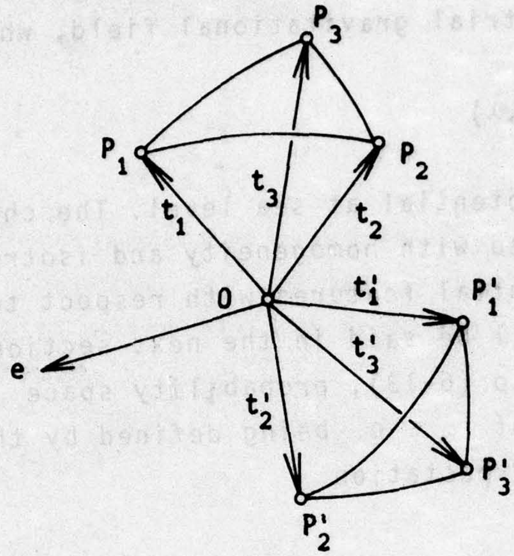

Figure 7. Rotation of a configuration

Each of the vectors $t_1, t_2, t_3 \;;\; t_1', t_2', t_3'$ can be obtained by rotating a fixed unit vector e (for which we may take, for instance, the unit vector of the X-axis) by a certain matrix $R(\omega_1), R(\omega_2), \ldots, R(\omega_3')$; we write $R(\omega_i)$ instead of R_{ω_i} to avoid two-level subscripts. Then, for $i = 1, 2, 3$,

$$t_i = R(\omega_i)e \;, \quad t_i' = R(\omega_i')e \;. \tag{6-18}$$

Combining (6-17) and (6-18) we have

$$t_i' = R(\omega_i')e = R_1 R(\omega_i)e$$

or

$$R(\omega_i') = R_1 R(\omega_i) \;. \tag{6-19}$$

Here $i = 1, 2, 3$, but clearly the configuration rotated by R_1 can have any number of points.

Thus, multiplying, from the left, a set of rotation matrices $R(\omega_i)$, $i = 1, 2, 3, \ldots,$ by a fixed matrix R_1 preserves the configuration. The geometrical configuration is invariant with respect to left multiplication. Similarly we may show the invariance of geometry with respect to right multiplication.

Homogeneity and isotropy imply that the essential properties depend only on the geometrical configuration. Therefore, also the probability measure must be invariant with respect to right and left multiplication. It can be shown that for a compact group such as the rotation group, there is essentially (apart from a constant factor) only one group measure that is both right and left invariant (Smirnow, III 1, § 89). Such an invariant volume element in rotation group space is

$$dV = \sin\Theta \, d\Theta \, d\Lambda \, dA \; ; \qquad (6\text{-}20)$$

this will be proved later in this section. The total volume of group space is, by (6-3),

$$V = \int_{\Lambda=0}^{2\pi} \int_{\Theta=0}^{\pi} \int_{A=0}^{2\pi} \sin\Theta \, d\Theta \, d\Lambda \, dA = 8\pi^2 \; ,$$

so that

$$d\Omega = \frac{1}{8\pi^2} \sin\Theta \, d\Theta \, d\Lambda \, dA \qquad (6\text{-}21)$$

is the desired element of probability measure.

Now we are ready to attack the computation of expectations and covariances. The expectation $E\{f(t,\omega)\}$ becomes

$$E\{f(t,\omega)\} = \iiint_{\Omega} f(R_\omega t)\,d\Omega$$

$$= \iiint_{\Omega} f(R_\omega R_{-\tau} t)\,d\Omega \qquad (6\text{-}22)$$

$$= \iiint_{\Omega} f(R_\omega e_Z)\,d\Omega \ ,$$

in view of right invariance and using (6-11). However, $R_\omega e_Z$ transforms the unit vector e_Z of the Z-axis into the unit vector of the z-axis, which has the spherical coordinates Θ and Λ (Fig.6). Hence,

$$f(R_\omega e_Z) = f(\Theta,\Lambda) \ , \qquad (6\text{-}23)$$

and (6-22) becomes

$$E\{f(t,\omega)\} = \frac{1}{8\pi^2} \int_{\Lambda=0}^{2\pi} \int_{\Theta=0}^{\pi} \int_{A=0}^{2\pi} f(\Theta,\Lambda) \sin\Theta \, d\Theta \, d\Lambda \, dA \ .$$

We integrate over A and replace Θ, Λ by θ, λ, respectively; obviously, the symbols for the integration variables is irrelevant. The result is

$$E\{f(t,\omega)\} = \frac{1}{4\pi} \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} f(\theta,\lambda) \sin\theta \, d\theta \, d\lambda \ , \qquad (6\text{-}24)$$

which is simply the average over $f(t)$ over the unit sphere. It is zero if $f(\theta,\lambda)$ contains no zero-degree spherical harmonic, which corresponds to our usual assumption that the stochastic process under consideration is centered.

Then, by (4-16), the covariance function becomes

$$C(t,u) = E\{f(t)f(u)\} \tag{6-25}$$

with

$$t = (\theta,\lambda) \ , \qquad u = (\theta',\lambda') \ . \tag{6-26}$$

More explicitly this is written

$$C(t,u) = \iiint_{\Omega} f(R_\omega t) f(R_\omega u) d\Omega \ , \tag{6-27}$$

which, because of right invariance, is equal to

$$C(t,u) = \iiint_{\Omega} f(R_\omega R_{-\tau} t) f(R_\omega R_{-\tau} u) d\Omega \ . \tag{6-28}$$

Now, by (6-11) and (6-23),

$$f(R_\omega R_{-\tau} t) = f(\Theta,\Lambda) \tag{6-29}$$

gives the value of f at a point P with spherical coordinates (Θ,Λ) , whereas

$$f(R_\omega R_{-\tau} u) = f(\Theta',\Lambda') \tag{6-30}$$

denotes the value of f at some point $Q = (\Theta',\Lambda')$ situated at the spherical distance ψ from P (Fig.8). That the spherical distance ψ between P and Q is equal to the spherical distance between the points t and u as given by (4-23) follows from the invariance of the configuration with respect to the rotation $R_\omega R_{-\tau}$. It is also easily seen that

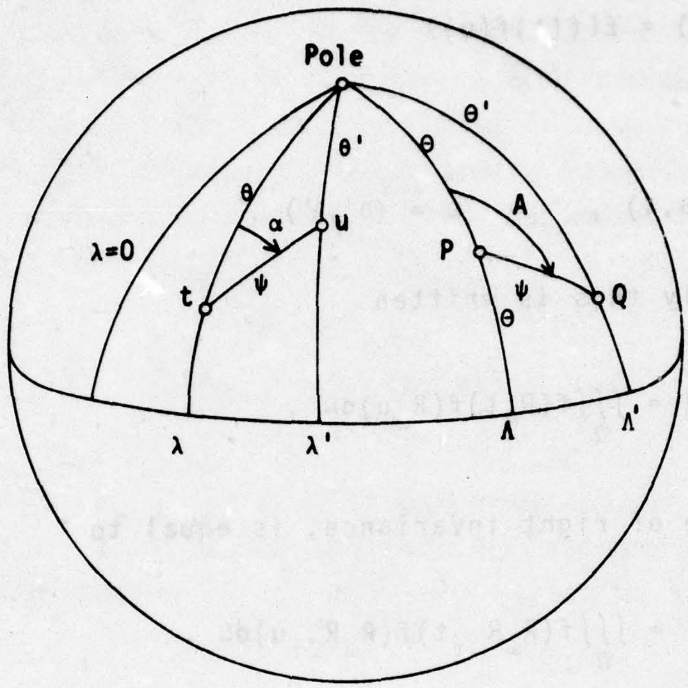

Figure 8. Invariance of the spherical distance ψ

if α is chosen as the azimuth from t to u, then A in (6-2) will be the azimuth from P to Q.

Thus (6-28) becomes

$$C(t,u) = \frac{1}{8\pi^2} \int_{\Lambda=0}^{2\pi} \int_{\Theta=0}^{\pi} \int_{A=0}^{2\pi} f(\Theta,\Lambda) f(\Theta',\Lambda') \sin\Theta \, d\Theta \, d\Lambda \, dA \, . \quad (6\text{-}31)$$

On replacing Θ, Λ, A by θ, λ, α we get

$$C(t,u) = \frac{1}{8\pi^2} \int_{\alpha=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} f(\theta,\lambda) f(\theta',\lambda') \sin\theta \, d\theta \, d\lambda \, d\alpha \, . \quad (6\text{-}32)$$

Formally, this is only a change in the symbols for the integration variables; but geometrically the meaning is now altered profoundly: a comparison with (4-25) shows that this is simply the average M , so that, by (4-26),

$$C(t,u) = E\{f(t,\omega)f(u,\omega)\} = M\{f(t)f(u)\} = \Gamma(\psi) \quad ; \qquad (6\text{-}33)$$

the true and the empirical covariance functions are identical.

Exactly as in the two-dimensional case, the reason for this identity is simply that probability space is made to coincide with rotation group space, so that the expectation E and the rotation group average M are identical; our present model is <u>trivially ergodic</u>. The present lengthy discussion has been made because the first brief presentation of this model in (Moritz, 1972, 1973, sec. 9) has proved to be too sketchy.

It is very interesting to study the stochastic behaviour of the spherical-harmonic coefficients $\bar{a}_{nm}(\omega)$ in the present model; the situation is less simple than in the two-dimensional analogue.

Specializing (4-10) for the present model we get for fully normalized harmonics

$$\begin{aligned}
\bar{a}_{nm}(\omega) &= \frac{1}{4\pi} \iint_\sigma f(t,\omega)\bar{R}_{nm}(t)d\sigma \\
&= \frac{1}{4\pi} \iint_\sigma f(R_\omega t)\bar{R}_{nm}(t)d\sigma \\
&= \frac{1}{4\pi} \iint_\sigma f(R_\omega R_{-\omega} t)\bar{R}_{nm}(R_{-\omega} t)d\sigma \\
&= \frac{1}{4\pi} \iint_\sigma f(t)\bar{R}_{nm}(R_{-\omega} t)d\sigma \ ,
\end{aligned} \qquad (6\text{-}34)$$

again because of the rotational invariance of the integral.

It is now well known how spherical harmonics transform under rotation. References to the numerous literature are given in (Aardom, 1969); we shall mainly follow (Courant and Hilbert, 1953, pp.535-545). The transformed Legendre harmonics $\bar{R}_{nm}$ are simply linear combinations of all $2n+1$ Legendre harmonics of the same degree n :

$$\bar{R}_{nm}(R_{-\omega}t) = \sum_{\ell=-n}^{n} A_{nm\ell}(\omega)\bar{R}_{n\ell}(t) ; \tag{6-35}$$

the coefficients $A_{nm\ell}$ in this linear combination are obviously functions of ω , that is, of the rotation parameters.

On substituting (6-35) and integrating we get

$$\bar{a}_{nm}(\omega) = \sum_{\ell=-n}^{n} A_{nm\ell}(\omega)\bar{a}_{n\ell} , \tag{6-36}$$

where

$$\bar{a}_{n\ell} = \frac{1}{4\pi} \iint_{\sigma} f(t)\bar{R}_{n\ell}(t)d\sigma \tag{6-37}$$

are the constant coefficients of the original function $f(t)$.

Eq. (6-36) is the three-dimensional analogue of (3-24) and (3-25). Unfortunately, the explicit expressions of $A_{nm\ell}(\omega)$ are rather complicated. So we shall not try by direct computation to verify the orthogonality of the $\bar{a}_{nm}(\omega)$ and the fact that they satisfy (5-1): this follows indirectly from the identity between the empirical and the true covariance function. However, we should like to point out the following interesting difference between the two-dimensional and the spatial case. In two dimensions, the stochastic orthogonality of the Fourier coefficients has been a consequence of the usual orthogonality of

the trigonometric functions over the unit circle, since the transformation coefficients are simply given by $\cos k\omega$ and $\sin k\omega$. In the present three-dimensional model, the stochastic orthogonality relations (4-18) are not a consequence of the orthogonality relations of usual spherical harmonics over the unit sphere, as might be expected by analogy. In fact, rotation group space is not the usual unit sphere, but is described by three parameters.

Now it is very curious, however, that the stochastic orthogonality relations in three dimensions are, in fact, a consequence of orthogonality relations for spherical harmonics, but in four-dimensional space! In fact, rotation group space may be identified with the three-dimensional "surface" of a unit sphere in four-dimensional space, and the rotation coefficients $A_{nm\ell}(\omega)$ are essentially spherical harmonics in this space; see below.

We shall now give some supplementary information on the mathematical structure of rotation group space. The reader not interested in these mathematical details may proceed directly to the next section.

Mathematical complements. According to (Smirnow, Vol.III/1, sec.90,p.271), the invariant integral in rotation group space has the form

$$\iiint\limits_V f(a_1, a_2, a_3)\frac{1}{\sqrt{1-a_1^2-a_2^2-a_3^2}}\, da_1 da_2 da_3$$

$$= \iiint\limits_V f(a_1, a_2, a_3)\frac{1}{a_o}\, da_1 da_2 da_3 \ , \qquad (6\text{-}38)$$

where a_o, a_1, a_2, a_3 denote four parameters related by

$$a_o^2 + a_1^2 + a_2^2 + a_3^2 = 1 \ . \qquad (6\text{-}39)$$

These four parameters, three of which are independent, may be used to describe a rotation. They are called Eulerian parameters (not to be confused with Eulerian angles) and are related to the representation of rotations by quaternions.

Geometrically the parameters a_1, a_2, a_3, a_4 may be interpreted as Cartesian coordinates in a four-dimensional auxiliary space; then (6-39) describes the unit sphere in this space.

The volume element

$$dV = \frac{1}{a_o} da_1 da_2 da_3 \tag{6-40}$$

in the invariant integral (6-38) is nothing else than the three-dimensional "surface" element of this four-dimensional unit sphere. In fact, the ordinary surface element of the unit sphere in three-dimensional space, expressed in terms of Cartesian coordinates xyz, is

$$d\sigma = \frac{1}{\sqrt{1-x^2-y^2}} dxdy = \frac{1}{z} dxdy \tag{6-41}$$

(cf. Smirnow, Vol.II, sec.62, p.176), of which (6-40) is the four-dimensional analogue.

The expression of the Eulerian parameters a_i in terms of the Eulerian angles Λ, Θ, $\Psi = -A$ is (Synge, 1960, p.19, eq.(II.7)):

$$\begin{aligned}
a_1 &= -\sin\frac{\Theta}{2}\sin\frac{\Lambda+A}{2}\,, \\
a_2 &= \sin\frac{\Theta}{2}\cos\frac{\Lambda+A}{2}\,, \\
a_3 &= \cos\frac{\Theta}{2}\sin\frac{\Lambda-A}{2}\,, \\
a_o &= \cos\frac{\Theta}{2}\cos\frac{\Lambda-A}{2}\,.
\end{aligned} \tag{6-42}$$

If we wish to express the group volume element (6-40) in terms of Λ, Θ, A we have

$$dV = \frac{1}{a_o} J \, d\Theta d\Lambda dA \,, \tag{6-43}$$

where

$$J = \begin{vmatrix} \frac{\partial a_1}{\partial \Lambda} & \frac{\partial a_2}{\partial \Lambda} & \frac{\partial a_3}{\partial \Lambda} \\ \frac{\partial a_1}{\partial \Theta} & \frac{\partial a_2}{\partial \Theta} & \frac{\partial a_3}{\partial \Theta} \\ \frac{\partial a_1}{\partial A} & \frac{\partial a_2}{\partial A} & \frac{\partial a_3}{\partial A} \end{vmatrix} \tag{6-44}$$

is the Jacobian determinant of the transformation (6-42).

On differentiation of (6-42) we find

$$\frac{\partial a_1}{\partial \Lambda} = -\frac{1}{2} a_2 \,, \quad \frac{\partial a_2}{\partial \Lambda} = \frac{1}{2} a_1 \,, \quad \frac{\partial a_3}{\partial \Lambda} = \frac{1}{2} a_o \,,$$

$$\frac{\partial a_1}{\partial A} = -\frac{1}{2} a_2 \,, \quad \frac{\partial a_2}{\partial A} = \frac{1}{2} a_1 \,, \quad \frac{\partial a_3}{\partial A} = -\frac{1}{2} a_o \,,$$

so that (6-44) becomes

$$J = \frac{1}{4} \begin{vmatrix} -a_2 & a_1 & a_o \\ b_1 & b_2 & b_3 \\ -a_2 & a_1 & -a_o \end{vmatrix} \,, \tag{6-45}$$

where we have put

$$b_i = \frac{\partial a_i}{\partial \Theta} \,. \tag{6-46}$$

We subtract the first row in (6-45) from the third and develop the determinant with respect to the third row thus modified. The result is

$$
\begin{aligned}
J &= -\frac{1}{2} a_o(-a_2 b_2 - a_1 b_1) \\
&= \frac{1}{2} a_o\left(a_1 \frac{\partial a_1}{\partial \Theta} + a_2 \frac{\partial a_2}{\partial \Theta}\right) \\
&= \frac{1}{4} a_o \sin\frac{\Theta}{2} \cos\frac{\Theta}{2} \\
&= \frac{1}{8} a_o \sin\Theta
\end{aligned}
\tag{6-47}
$$

Thus (6-43) becomes

$$dV = \frac{1}{8} \sin\Theta d\Theta d\Lambda dA \ , \tag{6-48}$$

which, apart from an irrelevant constant factor, coincides with (6-20). This was the first relation to be shown here.

Secondly, we briefly consider the transformation of spherical harmonics under rotation, following (Courant and Hilbert, 1953, pp.542-545). The form (eq.(148), loc.cit.) given there is

$$P_{n\ell}(\cos\theta')e^{i\ell\lambda'} = \sum_{r=-n}^{n} \frac{(n-r)!}{(n-\ell)!} S_{2n}^{\ell,r} P_{nr}(\cos\theta)e^{ir\lambda} \ . \tag{6-49}$$

It is slightly different from (6-35) but is essentially equivalent in view of

$$e^{ir\lambda} = \cos r\lambda + i \sin r\lambda \ . \tag{6-50}$$

For the transformation coefficients $S_{2n}^{\ell,r}$ there holds (<u>ibid</u>., p.544)

$$S_{2n}^{\ell,r} = v^{-2n} H_{2n}^{n+\ell,n+r}(a_o, a_1, a_2, a_3) , \qquad (6\text{-}51)$$

where

$$v^2 = a_o^2 + a_1^2 + a_2^2 + a_3^2 ; \qquad (6\text{-}52)$$

the functions H are harmonic polynomials in a_k (<u>ibid</u>., p.542):

$$\left(\frac{\partial^2}{\partial a_o^2} + \frac{\partial^2}{\partial a_1^2} + \frac{\partial^2}{\partial a_2^2} + \frac{\partial^2}{\partial a_3^2}\right) H_{2n}^{n+\ell,n+r} = 0 . \qquad (6\text{-}53)$$

The functions $S_{2n}^{\ell,r}$ are defined on the surface of the unit sphere (6-39) and form an orthogonal system. Courant expresses them in terms of three parameters $\rho, \sigma, \tau,$ which are related to our parameters Θ, Λ, A by

$$\tau = \frac{\Theta}{2} , \quad \rho = \frac{\Lambda - A}{2} , \quad \sigma = \frac{\Lambda + A}{2} . \qquad (6\text{-}54)$$

7. <u>Statistical Distributions in Rotation Group Space</u>

In the last section we have studied rotation group space as a probability space primarily with respect to covariances. The covariance theory of stochastic process is what is needed for linear least squares prediction and estimation problems; it can be treated without explicit reference to the underlying statistical distributions, of which only the moments of first order (mean values), and of second order (variances and covariances) are needed.

Even in least-squares prediction and collocation, however, the distributions of relevant quantities are required if we wish to perform statistical tests. Already to answer very elementary but meaningful questions we need distributions.

Such a question is, for instance: What is the average global frequency of a $1^{\circ} \times 1^{\circ}$ mean gravity anomaly situated between - 28 and - 36 mgal? This question is answered by the histogram of Fig. 9; the frequency is the number of $1^{\circ} \times 1^{\circ}$ anomalies having magnitude within a specified interval, divided by the total number of $1^{\circ} \times 1^{\circ}$ anomalies. Clearly, such a frequency can be considered as a measure of the probability that a $1^{\circ} \times 1^{\circ}$ mean anomaly lies between - 28 and - 36 mgal.

A similarly meaningful question would be: What is the probability that a $1^{\circ} \times 1^{\circ}$ mean Δg-value lies between - 28 and - 36 mgal and that the mean value of the geoidal height N for the same $1^{\circ} \times 1^{\circ}$ block lies between 25 and 30 meters?

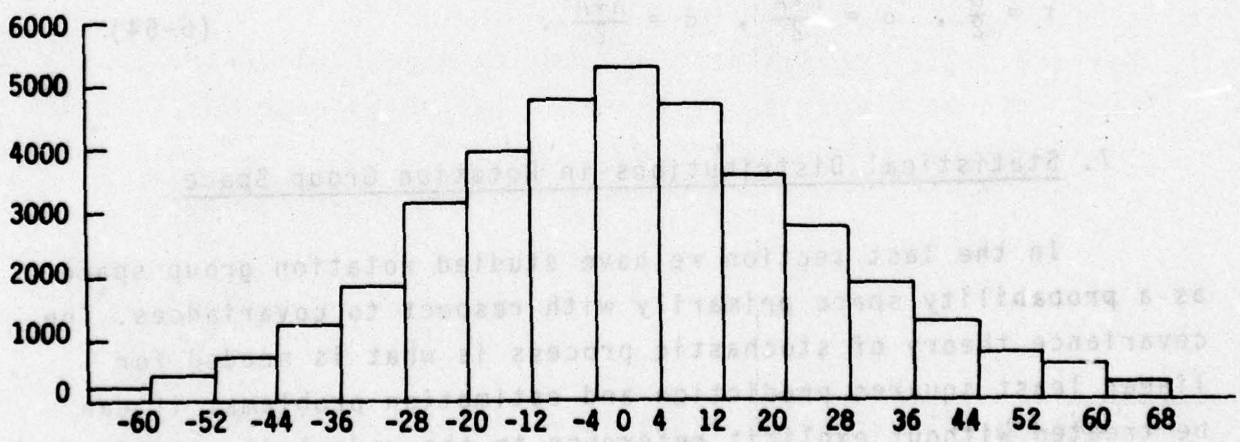

Figure 9. Number of $1^{\circ} \times 1^{\circ}$ mean gravity anomalies having magnitude within specified interval. After (Rapp,1977,p.5).

To answer such and related question, we must construct appropriate distribution functions for Δg, for Δg and N jointly, etc. The distribution density for Δg will be a continuous analogue of the histogram of Fig.9. To find an appropriate probability space, let us note that the number of relevant $1^\circ \times 1^\circ$ mean anomalies is counted regardless of the position, on the earth's surface, of the $1^\circ \times 1^\circ$ blocks under consideration. All positions on the sphere are treated equally: again we have homogeneity and isotropy. Thus, rotation group space is seen to be the proper probability space also as a basis for the mathematical description of statistical distributions.

The nature of a statistical distribution is best illustrated by the case of a function of one variable. Therefore, we shall again start with the group of rotations of the circle (the two-dimensional rotation group), which is parametrized by one variable ω, $0 \leqq \omega < 2\pi$. Probability space Ω is, therefore, the unit circle, and the element of probability measure is $\frac{1}{2\pi} d\omega$, which is clearly left and right invariant. Let the random function under consideration be denoted by $f(\omega)$.

We plot ω along the horizontal axis of a graph; then $f(\omega)$ is defined for $0 \leqq \omega < 2\pi$ (it could, of course, be continued periodically for other abscissas).

Then the distribution function $\Phi(x)$ is defined by

$$\Phi(x) = \text{Prob}\{f(\omega) < x\} \tag{7-1}$$

as the probability that $f(\omega)$ takes a value smaller than x. It is the measure of all values of ω for which $f(\omega) < x$; this measure is obviously a function of x. In the situation shown in Fig. 10, $f(\omega) < x$ if ω is contained in the interval AB or in the interval CD ; thus

$$\text{Prob}\{f(\omega) < x\} = \frac{1}{2\pi}(\overline{AB} + \overline{CD}) ,$$

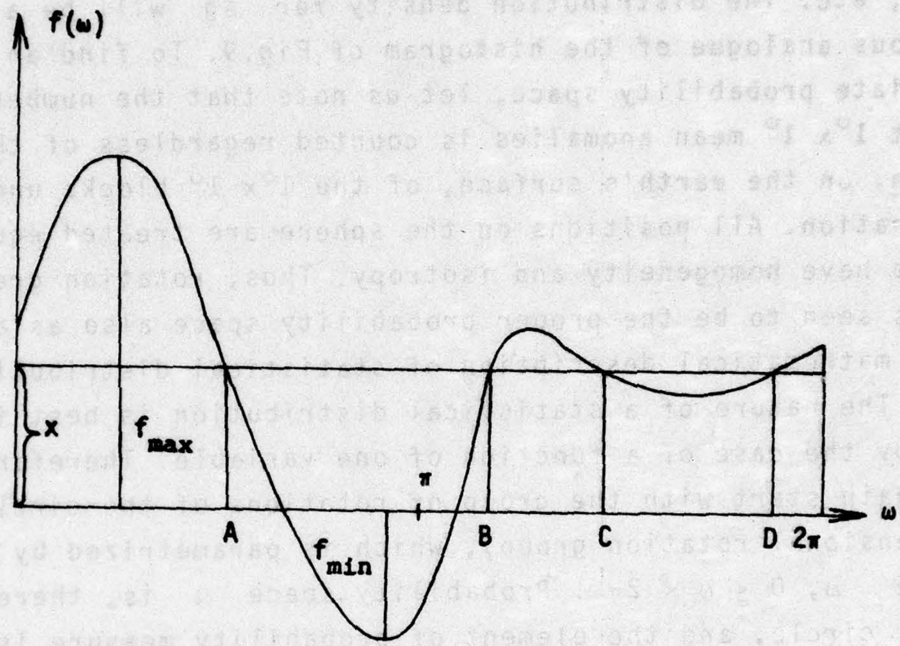

Figure 10. Distribution of a random function f(ω)

$\overline{AB}$ denoting the length of AB and the factor $1/2\pi$ serving to make the measure of the total interval from 0 to 2π equal to unity.

Generally we may write

$$\Phi(x) = \frac{1}{2\pi} \int_{f(\omega)<x} d\omega \ , \qquad (7\text{-}2)$$

the integral being extended over those points ω for which $f(\omega) < x$.

The derivative of (7-1) with respect to x gives the probability density

$$\phi(x) = \Phi'(x) = \frac{d\Phi}{dx} \ , \qquad (7\text{-}3)$$

which has an even more intuitive geometrical meaning. For a differential dx (operations with differentials are justified in the usual way) we have

$$\phi(x)dx = d\Phi(x) = \mathrm{Prob}\{x < f(\omega) < x + dx\} \ . \qquad (7\text{-}4)$$

According to Fig. 11, which represents the same function, $f(\omega)$ assumes a value between x and $x + dx$ if x lies in one of the small intervals A'A, BB', C'C, or DD', so that

$$d\Phi(x) = \frac{1}{2\pi}(\overline{A'A} + \overline{BB'} + \overline{C'C} + \overline{DD'}) \ , \qquad (7\text{-}5)$$

or

$$\phi(x) = \frac{1}{2\pi dx}(\overline{A'A} + \overline{BB'} + \overline{C'C} + \overline{DD'}) \ , \qquad (7\text{-}6)$$

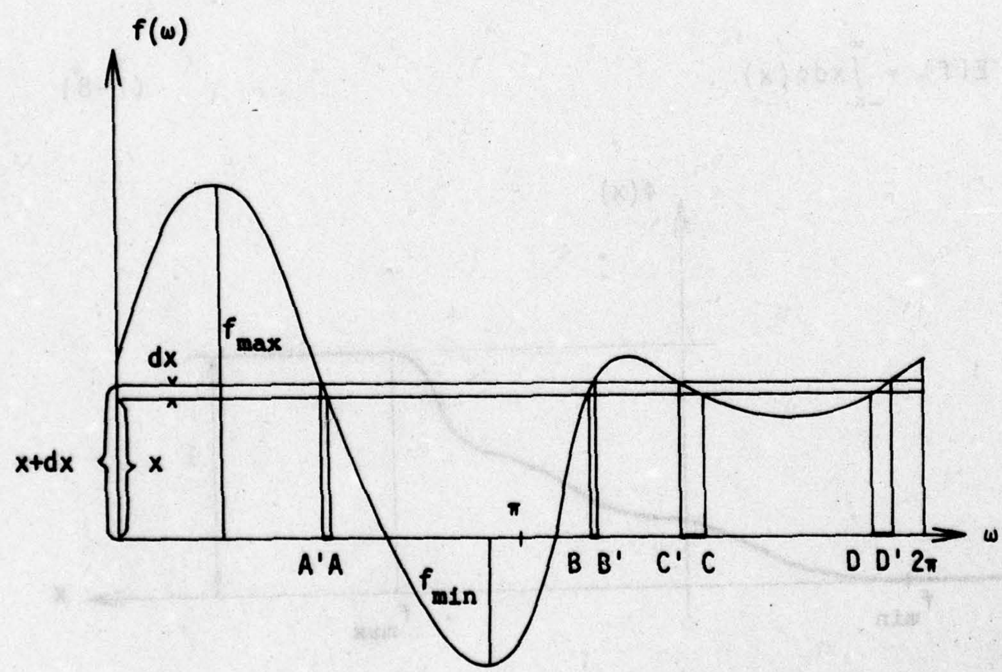

Figure 11. Definition of the distribution density

which gives an intuitive geometrical interpretation of the distribution density $\phi(x)$.

The distribution function $\Phi(x)$ is a monotone non-negative function of x , $-\infty < x < \infty$, as shown in Fig. 12. It is identically <u>zero</u> for $x < f_{min}$ (there is <u>no</u> ω for which $f(\omega) < f_{min}$) and identically <u>one</u> for $x > f_{max}$ (for <u>all</u> ω there is $f(\omega) < x$ if $x > f_{max}$) .

The statistical expectation E of the random variable f can now be computed in two ways: by means of the probability measure $d\omega/2\pi$:

$$E\{f\} = \frac{1}{2\pi} \int_0^{2\pi} f(\omega)d\omega \ , \tag{7-7}$$

or by means of the distribution function

$$E\{f\} = \int_{-\infty}^{\infty} x d\Phi(x) \ . \tag{7-8}$$

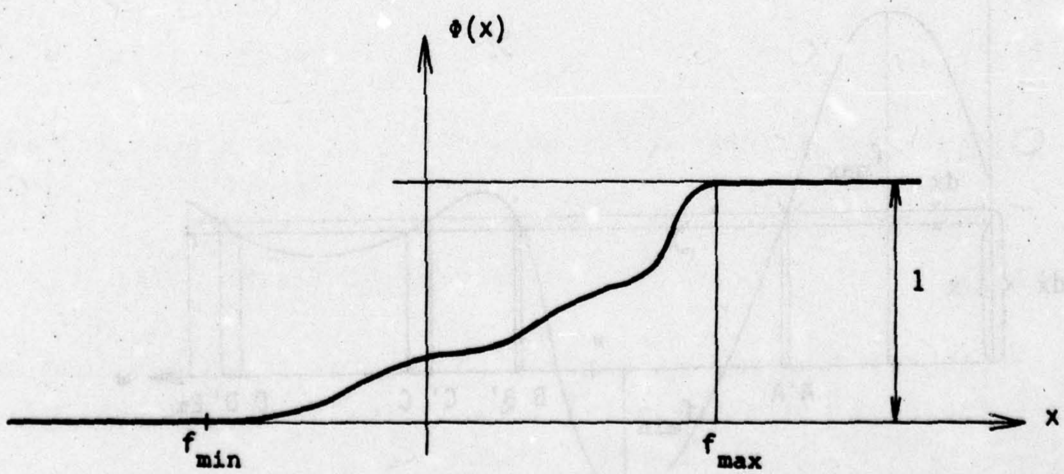

Figure 12. A distribution function

Geometrically, both expressions give the area under the curve $f(\omega)$ in Fig.10; (7-7) corresponds to the Riemann partitioning and (7-8) corresponds to the Lebesgue partitioning of the same definite integral; therefore (7-7) and (7-8) are identical; cf. (Feller, 1966, p.115-116) and (Kolmogorov and Fomin, 1970, p.293).

Let us, finally, consider the stochastic process (3-12),

$$f(t,\omega) = f(t+\omega) \ . \qquad (7\text{-}9)$$

In view of the rotational invariance, the distribution function of $f(t+\omega)$, for fixed t, is the same for all t, hence, is the same as for $t = 0$:

$$\mathrm{Prob}\{f(t+\omega) < x\} = \mathrm{Prob}\{f(\omega) < x\} \ . \qquad (7\text{-}10)$$

This is immediately seen from Fig. 11: replacing t by $t+\omega$ means only a translation of the figure as a whole to the right or left; the length of the intervals A'A, ... and hence (7-5) or (7-6), remaining unchanged.

What is more, we may also write

$$\Phi(x) = \mathrm{Meas}\{f(t) < x\} \ , \qquad (7\text{-}11)$$

using only the sample function $f(t)$ defined on the "space circle" $0 \leqq t < 2\pi$ with measure "Meas" defined by its element $dt/2\pi$, without any probabilistic interpretation; this follows immediately by replacing ω in (7-1) by t. This is formally very simple, but conceptually of fundamental importance: it shows that we may consistently work with one basic sample function $f(t)$ only and still avail ourselves of the formal advantages of probability theory.

<u>Distributions in three-dimensional rotation group space.</u> After these preliminaries we come to the geodetically relevant case of three-dimensional rotations. The basic ideas remain the same, though the notation is more cumbersome.

Let $f(\omega)$ be a real-valued random function; the argument is defined by (6-2). Then the distribution function $\Phi(x)$ of f is

$$\Phi(x) = \mathrm{Prob}\{f(\omega) < x\} . \tag{7-12}$$

It should be noted that x is a one-dimensional real variable, $-\infty < x < \infty$, though ω denotes a point in three-dimensional rotation group space Ω.

Consider now the random function (6-13),

$$f(t,\omega) = f(R_\omega t) \quad \text{with} \quad t = (\theta,\lambda) . \tag{7-13}$$

In view of the rotational invariance, the distribution function of

$$\Phi(x) = \mathrm{Prob}\{f(R_\omega t) < x\} \tag{7-14}$$

does not depend on t. Following the reasoning that leads from (6-22) to (6-24) we find that

$$\Phi(x) = \mathrm{Meas}\{f(\theta,\lambda) < x\} \tag{7-15}$$

The measure "Meas" is surface measure on the unit sphere, normalized by the factor $1/4\pi$; its element is, as usual,

$$\frac{1}{4\pi}\, d\sigma = \frac{1}{4\pi} \sin\theta\, d\theta\, d\lambda . \tag{7-16}$$

Just as in (6-24), there is no longer an explicit dependence on the azimuth variable A.

For the distribution density

$$\phi(x) = \Phi'(x) \tag{7-17}$$

we have again a geometrical interpretation (Fig.13). Draw the neighboring contour lines

$$f(\theta,\lambda) = x = \text{const.},$$

$$f(\theta,\lambda) = x + dx = \text{const.}$$

on the sphere; they will, in general, consist of several unconnected closed curves. Let the areas between these neighboring closed lines be denoted by dA_1, dA_2, dA_3, ... (hatched in Fig.13). Then

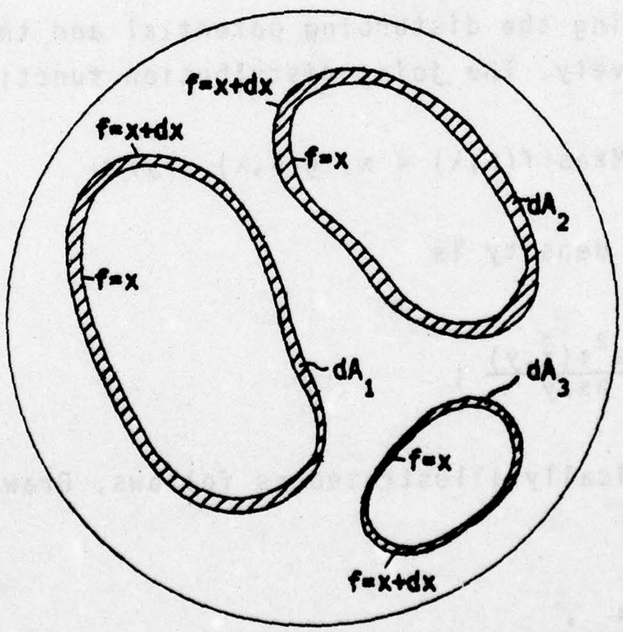

Figure 13. Geometrical interpretation of the distribution density

$$\phi(x)dx = \frac{1}{4\pi}(dA_1 + dA_2 + dA_3 + \ldots) \,. \tag{7-18}$$

The distribution function $\Phi(x)$ itself can be expressed in a form analogous to (7-2):

$$\Phi(x) = \frac{1}{4\pi} \iint_{f(\theta,\lambda)<x} \sin\theta d\theta d\lambda \,. \tag{7-19}$$

Another basic problem is the determination of the joint distribution of two functions f and g on the sphere, say, of

$$\begin{aligned} f(\theta,\lambda) &= T(\theta,\lambda) \,, \\ g(\theta,\lambda) &= \Delta g(\theta,\lambda) \,, \end{aligned} \tag{7-20}$$

T and Δg denoting the disturbing potential and the gravity anomaly, respectively. The joint distribution function is

$$\Phi(x,y) = \mathrm{Meas}\{f(\theta,\lambda) < x,\ g(\theta,\lambda) < y\} \,. \tag{7-21}$$

The corresponding density is

$$\phi(x,y) = \frac{\partial^2\Phi(x,y)}{\partial x \partial y} \,; \tag{7-22}$$

it may be geometrically illustrated as follows. Draw the contour lines

$$\begin{aligned} f(\theta,\lambda) &= x \,, \\ f(\theta,\lambda) &= x + dx \,, \end{aligned} \tag{7-23}$$

as well as the contour lines

$$g(\theta,\lambda) = y \ , \qquad\qquad (7\text{-}24)$$
$$g(\theta,\lambda) = y + dy$$

(Fig. 14). The ribbons formed in this way intersect in areas dA_1, dA_2, dA_3, ... (hatched in Fig. 14), and

$$\phi(x,y)dxdy = \frac{1}{4\pi}(dA_1 + dA_2 + dA_3 + \ldots) \ . \qquad (7\text{-}25)$$

A final example will indicate how an azimuth-dependent situation can be handled. Consider the problem of the joint distribution of gravity anomalies at two points that are at a spherical distance ψ apart:

$$\Phi(x,y) = \mathrm{Prob}\{f(t,\omega) < x, \ f(u,\omega) < y\} \qquad (7\text{-}26)$$

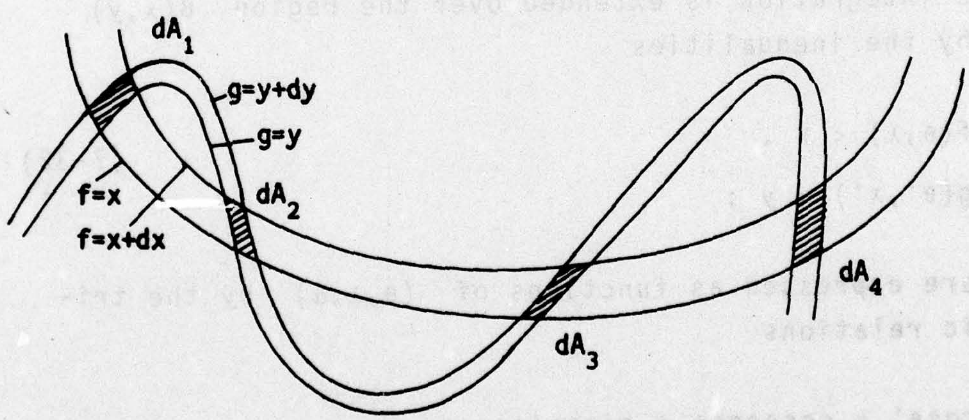

Figure 14. Joint distributions

where

$$\psi = \text{angle}(t,u) = \text{const.} \qquad (7\text{-}27)$$

We have

$$t = (\theta,\lambda) \ , \qquad (7\text{-}28)$$

$$u = (\theta',\lambda') \ , \qquad (7\text{-}29)$$

where the condition (7-27) can be written in the form

$$\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda'-\lambda) = \cos\psi = \text{const.} \qquad (7\text{-}30)$$

Then (7-26) can be expressed as the integral

$$\Phi(x,y) = \frac{1}{8\pi^2} \iiint_{B(x,y)} \sin\theta d\theta d\lambda d\alpha \ , \qquad (7\text{-}31)$$

where the integration is extended over the region $B(x,y)$ defined by the inequalities

$$\begin{aligned} &f(\theta,\lambda) < x \ , \\ &g(\theta',\lambda') < y \ ; \end{aligned} \qquad (7\text{-}32)$$

θ', λ' are expressed as functions of (θ,λ,α) by the trigonometric relations

$$\begin{aligned} &\cos\theta' = \cos\theta\cos\psi + \sin\theta\sin\psi\cos\alpha \ , \\ &\sin(\lambda'-\lambda) = \sin\psi\sin\alpha/\sin\theta' , \end{aligned} \qquad (7\text{-}33)$$

which follow from the spherical triangle of Fig. 15. The integral (7-31) is analogous to (7-2) and (7-19); the probability measure $\sin\theta d\theta d\Lambda dA$ has been replaced by "spatial" (surface plus azimuth)

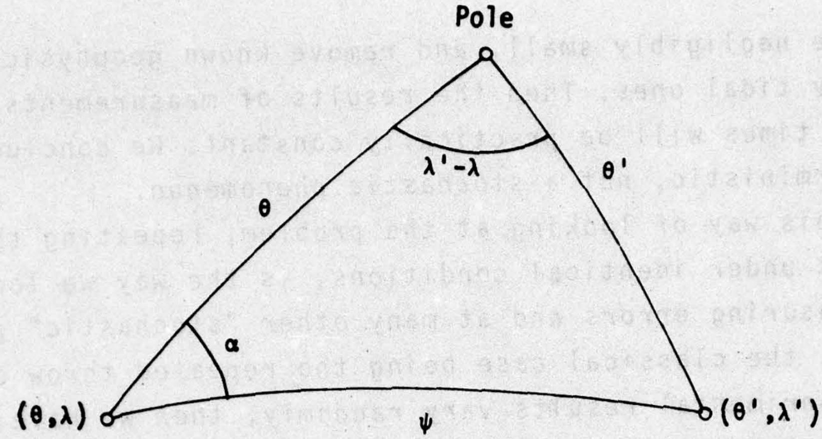

Figure 15. The basic spherical triangle

measure $\sin\theta d\theta d\lambda d\alpha$ in the same way as (6-31) has been replaced by (6-32).

These three examples illustrate the basic principles of the determination of single and joint distributions. Other cases can be handled similarly. In any case, we can operate with "spatial" functions $f(\theta,\lambda)$, $g(\theta,\lambda)$, ... only.

In practice, the functions $f(\theta,\lambda)$ are usually represented by discrete mean values (say, 5'x 5' or 1° x 1° block averages), and the integrations are to be replaced by sums.

8. The Meaning of Statistics in Collocation

Are gravity anomalies a stochastic phenomenon? There are different answers to this question.

To get one answer, consider gravity at one observation station and observe it repeatedly. Assume that the measuring

errors are negligibly small, and remove known geophysical effects, especially tidal ones. Then the results of measurements at different times will be practically constant. We conclude: gravity is a deterministic, not a stochastic phenomenon.

This way of looking at the problem, repeating the same experiment under identical conditions, is the way we look at random measuring errors and at many other "stochastic" physical phenomena, the classical case being the repeated throw of a die. If the experimental results vary randomly, then we have a genuinely stochastic phenomenon. Under the assumption that the outcomes of the repeated experiment are independent of one another, we have the scheme of repeated trials, fundamental in probability theory.

There is, however, also another way of looking at the question of stochasticity of gravity anomalies. They are caused by mass anomalies, visible and invisible ones. These mass anomalies show some regular features, for instance, mountain chains extending in a regular fashion from north to south. After removing known irregularities, however, the residuals are rather irregular. It is difficult to recognize a regular pattern. We may say, with some justification, that the residual gravity field is caused by randomly distributed mass anomalies. In this sense, gravity anomalies (after subtraction of known regular trends) may be said to be random, perhaps even stochastical. The randomness exists here not with respect to time, as it was in the first case (measurements at the same point but at different times), but with respect to space (measurements at the same time but at different points). The random behavior is more or less independent of position on the sphere and of direction: it is homogeneous and isotropic.

Thus, the global anomalous gravitational field may be irregular enough to be considered as a realization (a sample function) of a stochastic process. Is this sufficient for saying that " the anomalous gravitational field is a stochastic process"?

Personally, I do not think so, because there is no other physical realization: there is only one Earth.

The situation might well be compared the problem of a global statistics of the human population. There is a temporal variation, but it is systematic (expansion) rather than random, and, not being associated with the Club of Rome, I shall not consider it here. But we have random variations from one human individual to the other. There are regular trends--color of skin, political and religious beliefs--but there are genuinely irregular features left, distributed over the human population and hence over the earth's surface. This is not completely unlike the surface distribution of gravity anomalies (although the analogy, if pushed too hard, quickly becomes nonsense).

Is it permitted to study the global population statistics at a given time, to calculate various statistical distributions? Every one will answer this question in the affirmative, although there is only one global population. All statistical distributions are simply calculated on the basis of this population.

I think the statistics of the gravitational field must be handled similarly. We simply must take seriously the fact that there is only one field, and compute the whole statistics from this one field only.

The appropriate mathematical apparatus for studying the "second-order statistics" (variances and covariances) of the gravitational field is thus Norbert Wiener's (1930) "covariance analysis of individual functions" (Doob, 1949, sec.1). This model is implicit in almost all geodetic work in this field (Kaula, 1959, 1967; Heiskanen and Moritz, 1967); explicitly it was formulated in (Moritz, 1973, sec.8). It essentially uses the idea of homogeneity and isotropy. For the sphere, homogeneity and isotropy really form a single compound notion, namely, invariance under rotations (in contrast to the plane, where homogeneity, invariance under translation, and isotropy, invariance under rotation, are separate notions!); this motivates

the introduction of the three-dimensional rotation group.

The present report attempts to extend Wiener's idea beyond a second-order theory, in such a way as to obtain a complete statistical theory including statistical distributions. This has been done in the last section; it has been seen that also distributions can, in fact, be obtained from one given function only.

Formally, this theory can also be interpreted within the framework of stochastic processes, as our ergodic Second Model. This is, of course, independent of the question whether the anomalous gravitational field is "really" a stochastic process in some physical sense. Probability theory simply serves to provide a convenient mathematical formalism. In this sense, formal probabilistic techniques have been successfully applied not only in analytical mechanics with a large number of particles (Khinchin, 1949), but even in analysis and number theory (Kac, 1959a,b).

The problem whether and in which respect the anomalous gravitational field is a "genuinely stochastic phenomenon" will be answered differently by different people, depending on their scientific outlooks. Even with respect to the philosophical meaning of "probability" and "stochastic phenomenon" there are many different, even quite opposite, opinions, as is seen by comparing books such as (Gnedenko, 1967, and (de Finetti, 1972); still, the mathematical formalism is the same.

Similarly, the mathematical formalism, proposed here as a statistical background of collocation, is independent of how serious we take the stochastic character of the gravitational field. Even if we rigorously deny this stochasticity, we can still accept the formal statistical analysis presented here: we then have "statistics without stochastics".

The statistics of measuring errors can be incorporated without problems, in the way described in (Moritz, 1973, secs.8 and 9): the combined phase space is the Cartesian product of rotation group space and of the probability space of the

measuring errors; a combined distribution function is simply the product of the distribution function of the field quantities under consideration and the distribution function of their measuring errors; and the averaging operator to be used is the (commutative) product of the rotation group average M and of the statistical expection E referring to the probability space of the measuring errors. If we limit ourselves to a second-order theory, then the approach of Sansò (1978) is logically particularly satisfactory.

If the approach of sections 6 and 7 is accepted, then a detailed theory of statistical distributions for geodetically relevant quantities, such as gravity anomalies, geoidal heights, and deflections of the vertical, could be developed and applied to the statistical testing of the results of least-squares collocation.

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